Boundary Control of PDEs:

A Course on Backstepping Designs

class slides

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Introduction

- *Fluid flows* in aerodynamics and propulsion applications;

  plasmas in lasers, fusion reactors, and hypersonic vehicles;

  liquid metals in cooling systems for tokamaks and computers, as well as in welding and metal casting processes;

  acoustic waves, water waves in irrigation systems...

- *Flexible structures* in civil engineering, aircraft wings and helicopter rotors, astronomical telescopes, and in nanotechnology devices like the atomic force microscope...

- *Electromagnetic* waves and quantum mechanical systems...

- *Waves* and “ripple” instabilities in thin film manufacturing and in flame dynamics...

- *Chemical* processes in process industries and in internal combustion engines...

Unfortunately, even “toy” PDE control problems like heat and wave equations (neither of which is unstable) require some background in functional analysis.

Courses in control of PDEs rare in engineering programs.

This course: methods which are easy to understand, minimal background beyond calculus.

Boundary Control

Two PDE control settings:

- <u>"in domain"</u> control (actuation penetrates inside the domain of the PDE system or is evenly distributed everywhere in the domain, likewise with sensing);

- <u>"boundary"</u> control (actuation and sensing are only through the boundary conditions).

Boundary control physically more realistic because actuation and sensing are non-intrusive (think, fluid flow where actuation is from the walls).*

*"Body force" actuation of electromagnetic type is also possible but it has low control authority and its spatial distribution typically has a pattern that favors the near-wall region.

Boundary control harder problem, because the “input operator” (the analog of the B matrix in the LTI finite dimensional model $\dot{x} = Ax + Bu$) and the output operator (the analog of the C matrix in $y = Cx$) are unbounded operators.

Most books on control of PDEs either don't cover boundary control or dedicate only small fractions of their coverage to boundary control.

This course is devoted exclusively to boundary control.

Backstepping

A particular approach to stabilization of dynamic systems with "triangular" structure.

Wildly successful in the area of nonlinear control since

[KKK] Krstic, Kanellakopoulos, Kokotovic
Nonlinear and Adaptive Control Design, 1995.

Other methods:

Optimal control for PDEs requires sol'n of operator Riccati equations (nonlinear and infinite-dimensional algebraic eqns).

Pole placement pursues precise assignment of a finite subset of the PDE's eigenvalues and requires model reduction.

Instead, backstepping achieves Lyapunov stabilization by transforming the system into a stable "target system."

A Short List of Other Books on Control of PDEs

- R. F. CURTAIN AND H. J. ZWART, *An Introduction to Infinite Dimensional Linear Systems Theory*, Springer-Verlag, 1995.
- I. LASIECKA, R. TRIGGIANI, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, Cambridge Univ. Press, 2000.
- A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR AND S. K. MITTER, *Representation and control of infinite–dimensional systems*, Birkhauser, 2006.
- Z. H. LUO, B. Z. GUO, AND O. MORGUL, *Stability and Stabilization of Infinite Dimensional Systems with Applications*, Springer Verlag, 1999.
- J. E. LAGNESE, *Boundary stabilization of thin plates*, SIAM, 1989.
- P. CHRISTOFIDES, *Nonlinear and Robust Control of Partial Differential Equation Systems: Methods and Applications to Transport-Reaction Processes*, Boston: Birkhäuser, 2001.

The Role of Model Reduction

Plays an important role in most methods for control design for PDEs.

They extract a finite dimensional subsystem to be controlled, while showing robustness to neglecting the remaining infinite dimensional dynamics in the design.

Backstepping does not employ model reduction—none is needed, except at the implementation stage.

Control Objectives for PDE Systems

- *Performance improvement*—for stable systems, optimal control.
- *Stabilization*—this course deals almost exclusively w/ unstable plants.
- *Trajectory tracking*—requires stabilizing fbk plus sol'n to trajectory generation probl.
- *Trajectory generation/motion planning*—towards the end of the course.

Classes of PDEs and Benchmark PDEs Dealt With in the Course

In contrast to ODEs, no general methodology for PDEs.

Two basic categories of PDEs studied in textbooks: *parabolic* and *hyperbolic* PDEs, with standard examples being heat and wave equations.

Many more categories.

Categorization of PDEs studied in the course

	∂_t	∂_{tt}
∂_x	transport PDEs, delays	
∂_{xx}	parabolic PDEs, reaction-advection-diffusion systems	hyperbolic PDEs, wave equations
∂_{xxx}	Korteweg-de Vries	
∂_{xxxx}	Kuramoto-Sivashinsky and Navier-Stokes (Orr-Sommerfeld form)	Euler-Bernoulli and shear beams, Schrodinger, Ginzburg-Landau

Timoshenko beam model has four derivatives in both time and space.

Also, complex-valued PDEs (with complex coefficients): Schrodinger and Ginzburg-Landau eqns. They "look" like parabolic PDEs, but behave like oscillatory, hyperbolic PDEs. Schrodinger equivalent to the Euler-Bernoulli beam PDE.

Choices of Boundary Controls

Thermal: actuate heat flux or temperature.

Structural: actuate beam's boundary position, or force, or angle, or moment.

Mathematical choices of boundary control:

Dirichlet control $u(1,t)$—actuate value of a function at boundary

Neumann control $u_x(1,t)$—actuate slope of a function at boundary

The Domain Dimension—1D, 2D, and 3D

PDE control complex enough in 1D: string, acoustic duct, beam, chemical tubular reactor, etc.

Can have finitely- and even infinitely-many unstable eigenvalues.

Some PDEs evolve in 2D and 3D but are dominated by phenomena evolving in one coordinate direction (while the phenomena in the other directions are stable and slow).

Some PDEs are genuinely 3D: Navier-Stokes.

See the companion book:

> Vazquez and Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flows*, Birkhauser, 2007.

Domain Shape in 2D and 3D

Rectangle or annulus much more readily tractable than a problem where the domain has an “amorphous/wiggly” shape.

Beware: literature abounds with abstract control methods for 2D and 3D PDE systems on general domains, where the complexities are hidden behind neatly written Riccati eqns.

Genuinely 2D or 3D systems, particularly if unstable and on oddly shaped domains (e.g., turbulent fluids in 3D around irregularly shaped bodies), truly require millions of differential equations to simulate and tens of thousands of equations to do control design for them.

Reasonable set up: boundary control of an endpoint of a line interval; edge of a rectangle; side of e parallelepiped.
(Dimension of actuation domain lower by one than dimension of PDE domain.)

Observers

Observer design using boundary sensing, dual to full-state fbk boundary control design.

Observer error system is exponentially stabilized.

Separation principle holds.

Adaptive Control of PDEs

Parameter estimators—system *identifiers*—for PDEs.

Unstable PDEs with unknown parameters controlled using parameter estimators supplied by identifiers and using state estimators supplied by adaptive observers.

See the companion book

> Smyshlyaev and Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

Nonlinear PDEs

At present, virtually no methods exist for boundary control of nonlinear PDEs.

Several results are available that apply to nonlinear PDEs that are neutrally stable and where the nonlinearity plays no destabilizing role.

No advanced control designs exist for broad classes of nonlinear PDEs that are open-loop unstable and where a sophisticated control Lyapunov function of non-quadratic type needs to be constructed to achieve closed-loop stability.

Though the focus of the course is on linear PDEs, we introduce basic ideas for stabilization of nonlinear PDEs at the end.

Delay Systems

A special class of ODE/PDE systems.

Delay is a transport PDE. (One derivative in space and one in time. First-order hyperbolic.)

Specialized books by Gu, Michiels, Niculescu.

A book focused on input delays, nonlinear plants, and unknown delays:

> M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

Organization of the Course

1. Basic Lyapunov stability ideas for PDEs. Backstepping transforms a PDE into a desirable "target PDE" within the same class. General Lyapunov thms for PDEs not very useful. We learn how to calculate *stability estimates* for a basic stable PDE and highlight the roles of spatial norms (L_2, H_1, and so on), the role of the Poincare, Agmon, and Sobolev inequalities, the role of integration by parts in Lyapunov calculations, and the distinction between energy boundedness and pointwise (in space) boundedness.

2. Eigenvalues, eigenfunctions, and basics of finding solutions of PDEs analytically.

3. Backstepping method. Our main "tutorial tool" is the reaction-diffusion PDE example

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t)\,,$$

on the spatial interval $x \in (0,1)$, with one uncontrolled boundary condition at $x = 0$,

$$u(0,t) = 0$$

and with a control applied through Dirichlet boundary actuation at $x = 1$.

4. Observer design. Develop a dual of backstepping for finding observer gain functions. Use reaction-diffusion PDE as an example.

5. Schrodinger and Ginzburg-Landau PDEs. Complex-valued but a backstepping design for parabolic PDEs easily extended. GL models vortex shedding.

6. Hyperbolic and “hyperbolic-like” equations—*wave equations, beams, transport equations, and delay equations*.

7. “Exotic” PDEs, with just one time derivative but with three and even four spatial derivatives—Kuramoto-Sivashinsky and Korteweg-de Vries eqns.

8. 3D Navier-Stokes eqn at high Reynolds number.

9. Motion planning/trajectory generation for PDEs. For example, how to find the time function for the input force for one end of a flexible beam to produce precisely the desired time-varying motion with the tip of the free end of the beam.

10. <u>Adaptive control</u> for parametrically uncertain PDEs.

11. <u>Nonlinear PDEs</u>.

Why We Don’t State Theorems

Focus on tools that allow to solve many problems, rather than on developing complete theorem statements for a few problems.

Want to move fast and cover many classes of PDEs and control/estimation topics.

Want to maintain physical intuition.

Want to make the material accessible to any control engineering grad student.

Focus on Unstable PDEs in 1D and Feedback Design Challenges

Unstable parabolic and hyperbolic PDEs in 1D with terms causing instability unmatched by the boundary control.

Feedback design challenges greater than the existence/uniqueness challenges, which are well addressed in analysis-oriented PDE books.

The Main Idea of Backstepping Control

Backstepping is a robust[†] extension of the "feedback linearization" approach for nonlinear finite-dimensional systems.

[†]Backstepping provides design tools that endow the controller with robustness to uncertain parameters and functional uncertainties in the plant nonlinearities, and robustness to external disturbances, robustness to other forms of modeling errors.

Feedback linearization entails two steps:

1. Construction of an *invertible change of variables* such that the system appears as linear in the new variables, except for a nonlinearity which is “in the span” of the control input vector;

2. *Cancellation of the nonlinearity*[‡] and the assignment of desirable linear exponentially stable dynamics on the closed-loop system.

[‡]In contrast to the standard feedback linearization, backstepping allows the flexibility to not necessarily cancel the nonlinearity. A nonlinearity may be kept if it is useful or it may be dominated (rather than cancelled non-robustly) if it is potentially harmful and uncertain.

Backstepping for PDEs:

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback, as in feedback linearization.

3. Find the state transformation as *identity minus a Volterra operator* (in x).
   Volterra operator = integral operator from 0 up to x (rather than from 0 to 1).
   Volterra transformation is “triangular” or “spatially causal.”

4. Obtain boundary feedback from the Volterra transformation. The transformation alone cannot eliminate the undesirable terms, but the transformantion brings them to the boundary, so control can cancel them.

Gain fcn of boundary controller = kernel of Volterra transformation.

Volterra kernel satisfies a *linear* PDE.

Backstepping is not “one-size-fits-all.” Requires structure-specific effort by designer.

Reward: elegant controller, clear closed-loop behavior.

Unique to This Course—Elements of Adaptive and Nonlinear Designs for PDEs

Prior to backstepping, state-of-the-art in *adaptive and nonlinear control* for PDEs comparable to the state-of-the-art for ODEs in the 1960s.

A wide range of PDE structures with nonlinearities, unknown parameters, and boundary control require backstepping.

Origins of This Course

Developed out of research results and papers by the instructor and his PhD students.

First taught as MAE 287 Distributed Parameter Systems at University of California, San Diego, in Fall 2005.

Lyapunov Stability

Recall some basics of stability analysis for linear ODEs.

An ODE

$$\dot{z} = Az\,, \qquad z \in \mathbb{R}^n \tag{1}$$

is exponentially stable (e.s.) at $z = 0$ if $\exists M > 0$ (overshoot coeff.) and $\alpha > 0$ (decay rate) s.t.

$$\boxed{\|z(t)\| \leq M\mathrm{e}^{-\alpha t}\|z(0)\|, \quad \text{for all } t \geq 0} \tag{2}$$

$\|\cdot\|$ denotes one of the equivalent vector norms, e.g., the 2-norm.

This is a *definition* of stability. If all the eigenvalues of the matrix A have negative real parts, this guarantees e.s., but this test is not always practical.

An alternative (iff) test which is more useful in state-space/time-domain and robustness studies:

> $\forall$ positive definite $n \times n$ matrix Q, $\exists$ a positive definite and symmetric matrix P s.t.
>
> $$\boxed{PA + A^T P = -Q\,.} \tag{3}$$

Lyapunov function:

$$V = x^T P x\,, \qquad \text{positive definite} \tag{4}$$

$$\dot{V} = -x^T Q x\,, \qquad \text{negative definite}\,. \tag{5}$$

For PDEs, an (infinite-dimensional) operator equation like (3) is hard to solve.

Key question for PDEs: not Lyapunov functions but system norms!

In finite dimension, vector norms are "equivalent." No matter which norm $\|\cdot\|$ one uses in (2) (for example, the 2-norm, 1-norm, or ∞-norm) one gets e.s. in the sense of any other vector norm. What changes are the constants M and α in (2).

For PDEs, the state space is not a Euclidean space but a function space, and likewise, the state norm is not a vector norm but a function norm.

Unfortunately, norms on function spaces are not equivalent. Bounds on the state in terms of the L_1, L_2, or L_∞ norm in x do not follow from one another.

To make matters more complicated, other natural choices of state norms for PDEs exist which are not equivalent with L_p norms. Those are the *Sobolev* norms, examples of which are the H_1 and H_2 norms (not to be confused with Hardy space norms in robust control for ODE systems), which, roughly, are the L_2 norms of the first and second derivative, respectively, of the PDE state.

With such a variety of choice, dictated by idiosyncracies of the PDE classes, general Lyapunov stability theory for PDEs is hopeless, though some efforts are made in

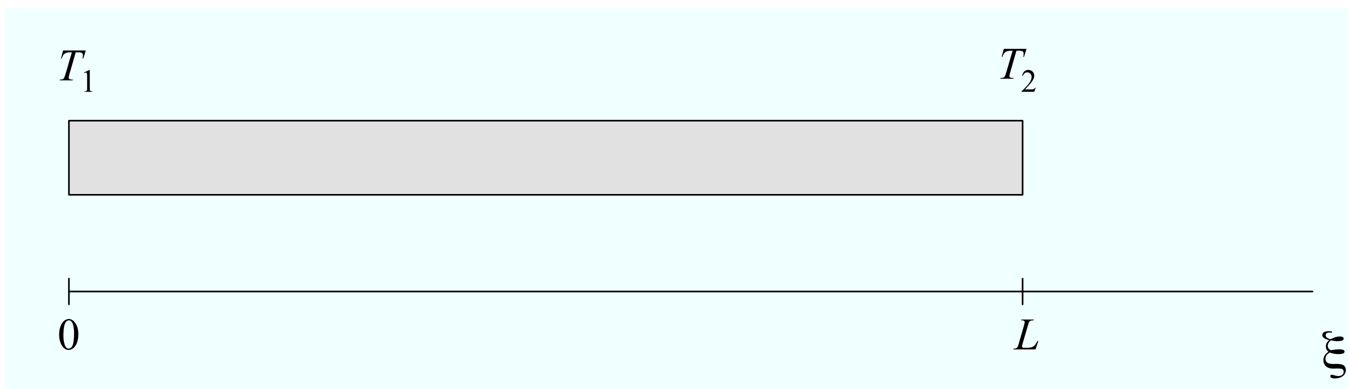
1. J. A. WALKER, *Dynamical Systems and Evolution Equations*, Plenum, 1980.

2. D. HENRY, *Geometric Theory of Semilinear Parabolic Equations*, Springer, 1993.

Instead, one is better off learning how to derive, from scratch, “energy estimates” (one’s own Lyapunov theorems) in different norms.

A Basic PDE Model

Before introducing stability concepts, we develop a basic “non-dimensionalized” PDE model, a 1D heat equation, which will help introduce the idea of energy estimates now, and be used as a target system for some backstepping designs later.



A thermally conducting rod.

The evolution of the temperature profile $T(\xi,\tau)$, as a function of the spatial variable ξ and time τ, is described by the heat equation[§]

$$\begin{aligned} T_\tau(\xi,\tau) &= \varepsilon T_{\xi\xi}(\xi,\tau)\,, \qquad x \in (0,L) && (6)\\ T(0,\tau) &= T_1\,, \qquad \text{left end of rod} && (7)\\ T(L,\tau) &= T_2\,, \qquad \text{right end of rod} && (8)\\ T(\xi,0) &= T_0(\xi)\,, \qquad \text{initial temperature distribution}\,. && (9) \end{aligned}$$

ε = thermal diffusivity

T_τ, $T_{\xi\xi}$ = partial derivatives with respect to time and space.

[§]While in physical heat conduction problems it is more appropriate to assume that the heat flux T_ξ is held constant at the boundaries (rather than the temperature T itself), for simplicity of our introductory exposition we proceed with the boundary conditions as in (7), (8) Our objective is to write this equation in nondimensional variables that describe the error between the unsteady temperature and the equilibrium profile of the temperature:

1. Scale ξ to normalize length:

$$x = \frac{\xi}{L}, \tag{10}$$

which gives

$$\begin{aligned} T_\tau(x,\tau) &= \frac{\varepsilon}{L^2} T_{xx}(x,\tau) && (11) \\ T(0,\tau) &= T_1 && (12) \\ T(1,\tau) &= T_2\,. && (13) \end{aligned}$$

2. Scale time to normalize thermal diffusivity:

$$t = \frac{\varepsilon}{L^2}\tau\,, \tag{14}$$

which gives

$$\begin{aligned} T_t(x,t) &= T_{xx}(x,t) && (15) \\ T(0,t) &= T_1 && (16) \\ T(1,t) &= T_2\,. && (17) \end{aligned}$$

3. Introduce new variable

$$w = T - T \tag{18}$$

where

$$T(x) = T_1 + x(T_2 - T_1)$$

is the steady-state profile and is found as a solution to the two-point boundary-value ODE

$$\begin{aligned} T''(x) &= 0 && (19)\\ T(0) &= T_1 && (20)\\ T(1) &= T_2 \,. && (21) \end{aligned}$$

We obtain

$$
\begin{aligned}
w_t &= w_{xx} && (22)\\
w(0) &= 0 && (23)\\
w(1) &= 0\,, && (24)
\end{aligned}
$$

with initial distribution $w_0 = w(x,0)$.

Note that here and throughout the rest of the course for compactness and ease of the presentation we drop the dependence on time and spatial variable where it does not lead to a confusion, i.e. by w, $w(0)$ we always mean $w(x,t)$, $w(0,t)$, respectively, unless specifically stated.

The following are the basic types of boundary conditions for PDEs in dimension one:

- Dirichlet: $\boxed{w(0) = 0}$ (fixed temperature at $x = 0$)

- Neumann: $\boxed{w_x(0) = 0}$ (fixed heat flux at $x = 0$)

- Robin (mixed): $\boxed{w_x(0) + qw(0) = 0}$

Lyapunov Analysis for a Heat Equation in Terms of 'L_2 Energy'

$$w_t = w_{xx} \tag{25}$$

$$w(0) = 0 \tag{26}$$

$$w(1) = 0\,. \tag{27}$$

Obviously stable for physical reasons and stability can also be shown by finding explic. soln.

But we want to learn a *method* for analyzing stability.

Lyapunov function candidate ("energy")[¶]

$$V(t) = \frac{1}{2}\int_0^1 w^2(x,t)\,dx = \frac{1}{2}\|w(t)\|^2 \tag{28}$$

where $\|\cdot\|$ denotes the L_2 norm of a function of x: $\boxed{\|w(t)\| = \left(\int_0^1 w(x,t)^2\,dx\right)^{1/2}.}$

[¶]Strictly speaking, this is a functional, but we refer to it simply as a "Lyapunov function."

Time derivative of V:

$$
\begin{aligned}
\dot{V} = \frac{dV}{dt} &= \int_0^1 w(x,t) w_t(x,t) dx && \text{(applying the chain rule)} \\
&= \int_0^1 w w_{xx} dx && \text{(from (25))} \\
&= \cancel{w w_x|_0^1} - \int_0^1 w_x^2 dx && \text{(integration by parts)} \\
&= -\int_0^1 w_x^2 dx. && (29)
\end{aligned}
$$

Since $\dot{V} \leq 0$, V is bounded. However, it is not clear if V goes to zero because (29) depends on w_x and not on w, so one cannot express the right hand side of (29) in terms of V.

Recall two useful inequalities:

Young's Inequality (special case)

$$\boxed{ab \leq \frac{\gamma}{2}a^2 + \frac{1}{2\gamma}b^2} \tag{30}$$

Cauchy-Schwartz Inequality

$$\boxed{\int_0^1 uw\,dx \leq \left(\int_0^1 u^2\,dx\right)^{1/2}\left(\int_0^1 w^2\,dx\right)^{1/2}} \tag{31}$$

The following lemma establishes the relationship between the L_2 norms of w and w_x.

Lemma 1 (Poincare Inequality) *For any w, continuously differentiable on $[0,1]$,*

$$\boxed{\begin{aligned} \int_0^1 w^2\,dx &\le 2w^2(1) + 4\int_0^1 w_x^2\,dx \\ \int_0^1 w^2\,dx &\le 2w^2(0) + 4\int_0^1 w_x^2\,dx \end{aligned}} \tag{32}$$

Remark 1 The inequalities (32) are conservative. A tighter version of (32) is

$$\int_0^1 w^2\,dx \le w^2(0) + \frac{8}{\pi^2}\int_0^1 w_x^2\,dx\,, \tag{33}$$

which is called “a variaton of Wirtinger’s inequality.” The proof of (33) is far more complicated than the proof of (32) and is given in the classical book on inequalities by Hardy, Littlewood, and Polya. When $\underline{w(0) = 0 \text{ or } w(1) = 0}$, one can even get $\boxed{\|w\| \le \frac{2}{\pi}\|w_x\|}$.

Proof.

$$\begin{aligned}
\int_0^1 w^2\,dx &= xw^2|_0^1 - 2\int_0^1 xww_x\,dx \quad \text{(integration by parts)} \\
&= w^2(1) - 2\int_0^1 xww_x\,dx \\
&\leq w^2(1) + \frac{1}{2}\int_0^1 w^2 dx + 2\int_0^1 x^2 w_x^2\,dx\,.
\end{aligned}$$

Subtracting the second term from both sides we get the first inequality in (32):

$$\begin{aligned}
\frac{1}{2}\int_0^1 w^2 dx &\leq w^2(1) + 2\int_0^1 x^2 w_x^2 dx \\
&\leq w^2(1) + 2\int_0^1 w_x^2 dx\,. \qquad (34)
\end{aligned}$$

The second inequality in (32) is obtained in a similar fashion. **QED**

We now return to

$$\dot{V} = -\int_0^1 w_x^2 dx\,.$$

Using Poincare inequality along with boundary conditions $w(0) = w(1) = 0$, we get

$$\dot{V} = -\int_0^1 w_x^2 dx \leq -\frac{1}{4}\int_0^1 w^2 \leq -\frac{1}{2}V \tag{35}$$

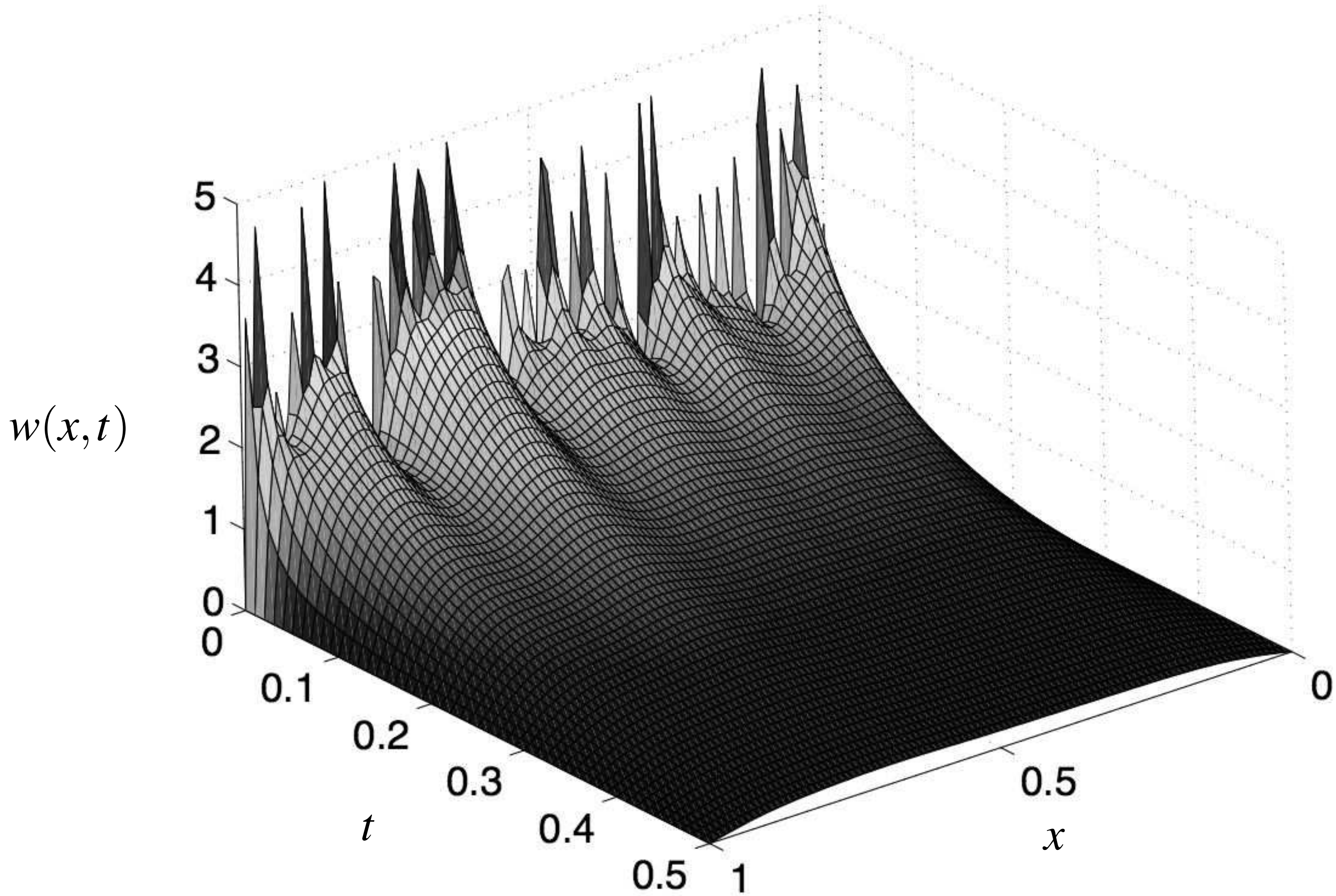
which, by the basic comparison principle for first order differential inequalities, implies that

$$V(t) \leq V(0)e^{-t/2}, \tag{36}$$

or

$$\boxed{\|w(t)\| \leq e^{-t/4}\|w_0\|} \tag{37}$$

Thus, the system (25)–(27) is exponentially stable in L_2.



Response of a heat equation to a non-smooth initial condition.

The “instant smoothing” effect is the characteristic feature of the diffusion operator that dominates the heat equation.

Pointwise-in-Space Boundedness and Stability in Higher Norms

We established that

$$\|w\| \to 0 \quad \text{as} \quad t \to \infty$$

but this does not imply that $w(x,t)$ goes to zero for each $x \in (0,1)$.

> Are there "unbounded spikes" for some x along the spatial domain (on a set of measure zero) which do not contribute to the L_2 norm?

It would be desirable to show that

$$\max_{x\in[0,1]} |w(x,t)| \leq e^{-\frac{t}{4}} \max_{x\in[0,1]} |w(x,0)|\,, \tag{38}$$

namely, stability in the spatial L_∞ norm. But this is possible only in some special cases and not worth our attention in a course that focuses on basic but generally applicable tools.

However, it is easy to show a more restrictive result than (38), given by

$$\max_{x\in[0,1]} |w(x,t)| \leq Ke^{-\frac{t}{2}}\|w_0\|_{H_1} \tag{39}$$

for some $K > 0$, where the H_1 norm is defined by

$$\boxed{\|w\|^2_{H_1} := \sqrt{\int_0^1 w^2 dx + \int_0^1 w_x^2 dx}} \tag{40}$$

Remark 2 The H_1 norm can be defined in different ways, the definition given above suits our needs. Note also that by using the Poincare inequality, it is possible to drop the first integral in (40) for most problems.

Before we proceed to prove (39), we need the following result.

Lemma 2 (Agmon's Inequality) *For a function $w \in H_1$, the following inequalities hold*

$$\boxed{\begin{aligned} \max_{x\in[0,1]} |w(x,t)|^2 &\leq w(0)^2 + 2\|w(t)\|\,\|w_x(t)\| \\ \max_{x\in[0,1]} |w(x,t)|^2 &\leq w(1)^2 + 2\|w(t)\|\,\|w_x(t)\| \end{aligned}} \tag{41}$$

Proof.

$$\begin{aligned} \int_0^x w w_x dx &= \int_0^x \partial_x \frac{1}{2} w^2 dx \\ &= \frac{1}{2} w^2 |_0^x \\ &= \frac{1}{2} w(x)^2 - \frac{1}{2} w(0)^2. \end{aligned} \tag{42}$$

Taking the absolute value on both sides and using the triangle inequality gives

$$\frac{1}{2}|w(x)^2| \leq \int_0^x |w||w_x| dx + \frac{1}{2} w(0)^2. \tag{43}$$

Using the fact that an integral of a positive function is an increasing function of its upper limit, we rewrite the last inequality as

$$|w(x)|^2 \leq w(0)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{44}$$

The right hand side of this inequality does not depend on x and therefore

$$\max_{x\in[0,1]} |w(x)|^2 \leq w(0)^2 + 2\int_0^1 |w(x)||w_x(x)|dx. \tag{45}$$

Using the Cauchy-Schwartz Inequality we get the first inequality of (41). The second inequality is obtained in a similar fashion. **QED**

The simplest way to prove $\max_{x\in[0,1]} |w(x,t)| \le Ke^{-\frac{t}{2}}\|w_0\|_{H_1}$ is to use the following Lyapunov function

$$V_1 = \frac{1}{2}\int_0^1 w^2\,dx + \frac{1}{2}\int_0^1 w_x^2\,dx\,. \tag{46}$$

The time derivative of (46) is given by

$$\begin{aligned}
\dot{V}_1 &\le -\|w_x\|^2 - \|w_{xx}\|^2 \le -\|w_x\|^2 \\
&\le -\frac{1}{2}\|w_x\|^2 - \frac{1}{2}\|w_x\|^2 \\
&\le -\frac{1}{8}\|w\|^2 - \frac{1}{2}\|w_x\|^2 \qquad \text{(using (35))} \\
&\le -\frac{1}{4}V_1\,.
\end{aligned}$$

Therefore,

$$\|w\|^2 + \|w_x\|^2 \le e^{-t/2}\left(\|w_0\|^2 + \|w_{0,x}\|^2\right), \tag{47}$$

and using Young's and Agmon's inequalities we get

$$\begin{aligned}
\max_{x\in[0,1]} |w(x,t)|^2 &\le 2\|w\|\|w_x\| \\
&\le \|w\|^2 + \|w_x\|^2 \\
&\le e^{-t/2}\left(\|w_0\|^2 + \|w_{x,0}\|^2\right). \qquad (48)
\end{aligned}$$

We have thus showed that

$$w(x,t) \to 0 \text{ as } t \to \infty$$

for all $x \in [0,1]$.

Example 1 Consider the *diffusion-* $\boxed{\textit{advection}}$ equation

$$
\begin{aligned}
w_t &= w_{xx} + \boxed{w_x} && (49)\\
w_x(0) &= 0 && (50)\\
w(1) &= 0\,. && (51)
\end{aligned}
$$

Using the Lyapunov function (28) we get

$$
\begin{aligned}
\dot{V} &= \int_0^1 w w_t dx \\
&= \int_0^1 w w_{xx} dx + \int_0^1 w w_x dx \\
&= w w_x|_0^1 - \int_0^1 w_x^2 dx + \boxed{\int_0^1 w w_x dx} && \text{(integration by parts)} \\
&= -\int_0^1 w_x^2 dx + \boxed{\frac{1}{2} w^2|_0^1} \\
&= -\int_0^1 w_x^2 dx + \boxed{\cancel{\frac{1}{2} w^2(0)} - \frac{1}{2} w^2(0)} \\
&= -\int_0^1 w_x^2 dx - \frac{1}{2} w^2(0)\,.
\end{aligned}
$$

Finally, using the Poincare inequality (32) we get

$$\dot{V} \leq -\frac{1}{4}\|w\|^2 \leq -\frac{1}{2}V\,, \tag{52}$$

proving the exponential stability in L_2 norm,

$$\|w(t)\| \leq \mathrm{e}^{-t/4}\|w_0\|.$$

Summary on Lyapunov function calculations so far

> It might appear that we are not constructing any non-trivial Lyapunov functions but only using the "diagonal" Lyapunov functions that do not involve any "cross-terms."

This is actually not the case with the remainder of the course. While so far we have studied only Lyapunov functions that are plain spatial norms of functions, in the sequel we are going to be constructing changes of variables for the PDE states. The Lyapunov functions will be employing the norms of the *transformed* state variables, which means that in the original PDE state our Lyapunov functions will be complex, sophisticated constructions that include 'non-diagonal' and 'cross-term' effects.

Homework

1. Prove the second inequalities in (32) and (41).

2. Consider the heat equation

$$w_t = w_{xx}$$

for $x \in (0,1)$ with the initial condition $w_0(x) = w(x,0)$ and boundary conditions

$$\begin{aligned} w_x(0) &= 0 \\ w_x(1) &= -\frac{1}{2}w(1). \end{aligned}$$

Show that

$$\|w(t)\| \leq e^{-\frac{t}{4}}\|w_0\|.$$

3. Consider the Burgers equation

$$w_t = w_{xx} - w w_x$$

for $x \in (0,1)$ with the initial condition $w_0(x) = w(x,0)$ and boundary conditions

$$\begin{aligned} w(0) &= 0 \\ w_x(1) &= -\frac{1}{6}\left(w(1) + w^3(1)\right). \end{aligned}$$

Show that

$$\|w(t)\| \leq e^{-\frac{t}{4}} \|w_0\|.$$

Hint: complete the squares.

Exact Solutions to PDEs

In general, seeking explicit solutions to partial differential equations is a hopeless pursuit.

But closed-form solutions can be found for some linear PDE systems with constant coefficients.

The solution does not only provide us with an exact formula for a given initial condition, but also gives insight into the spatial structure (smooth or ripply) and the temporal evolution (monotonic or oscillating) of the PDE.

Separation of Variables

Consider the reaction-diffusion equation

$$u_t = u_{xx} + \lambda u \tag{53}$$

with boundary conditions

$$u(0) = 0 \tag{54}$$

$$u(1) = 0 \tag{55}$$

and initial condition $u(x,0) = u_0(x)$.

The most frequently used method to obtain solutions to PDEs with constant coefficients is the method of separation of variables (the other common method employs Laplace transform).

Let us assume that the solution $u(x,t)$ can be written as

$$u(x,t) = X(x)T(t)\,. \tag{56}$$

If we substitute the solution (56) into the PDE (53), we get

$$X(x)\dot{T}(t) = X''(x)T(t) + \lambda X(x)T(t). \tag{57}$$

Gathering the like terms on the opposite sides yields

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + \lambda X(x)}{X(x)}. \tag{58}$$

Since the function on the left depends only on time and the function on the right depends only on the spatial variable, the equality can only hold if both functions are constant. Let us denote this constant by σ.

We then get two ODEs:

$$\dot{T} = \sigma T \tag{59}$$

with initial condition $T(0) = T_0$, and

$$X'' + (\lambda - \sigma)X = 0 \tag{60}$$

with boundary conditions $X(0) = X(1) = 0$ (they follow from the PDE boundary conditions).

The solution to (59) is given by

$$T = T_0 e^{\sigma t}. \tag{61}$$

The solution to (60) has the form

$$X(x) = A\sin(\sqrt{\lambda - \sigma}x) + B\cos(\sqrt{\lambda - \sigma}x), \tag{62}$$

where A and B are constants that should be determined from the boundary conditions.

We have:

$$X(0) = 0 \Rightarrow B = 0\,,$$
$$X(1) = 0 \Rightarrow A\sin(\sqrt{\lambda - \sigma}) = 0\,.$$

The last equality can only hold true if $\sqrt{\lambda - \sigma} = \pi n$ for $n = 0, 1, 2, \ldots$, so that

$$\sigma = \lambda - \pi^2 n^2, \qquad n = 0, 1, 2, \ldots \tag{63}$$

Substituting (61), (62) into (56) yields

$$u_n(x,t) = T_0 A_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x), \;\; n = 0, 1, 2, \ldots \tag{64}$$

For linear PDEs the sum of particular solutions is also a solution (the principle of superposition). Therefore the formal general solution of (53)–(55) is given by

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{(\lambda - \pi^2 n^2)t} \sin(\pi n x) \tag{65}$$

where $C_n = A_n T_0$.

To determine the constants C_n, let us set $t = 0$ in (65) and multiply both sides of the resulting equality with $\sin(\pi m x)$:

$$u_0(x)\sin(\pi m x) = \sin(\pi m x)\sum_{n=1}^{\infty} C_n \sin(\pi n x)\,. \tag{66}$$

Then, using the identity

$$\int_0^1 \sin(\pi m x)\sin(\pi n x)dx = \left\{ \begin{array}{c} 1/2 \;\; n = m \\ 0 \;\; n \neq m \end{array} \right\} \tag{67}$$

we get

$$C_n = \frac{1}{2}\int_0^1 u_0(x)\sin(\pi n x)dx\,. \tag{68}$$

Substituting this expression into (65), we get

$$\boxed{u(x,t) = 2\sum_{n=1}^{\infty} e^{(\lambda-\pi^2 n^2)t}\sin(\pi n x)\int_0^1 \sin(\pi n x)u_0(x)dx} \tag{69}$$

Even though we obtained this solution formally, it can be proved that this is indeed a well defined solution in a sense that it is unique, has continuous spatial derivatives up to a second order, and depends continuously on the initial data.

Let us look at the structure of this solution. It consists of the following elements:

- eigenvalues (all real): $\lambda - \pi^2 n^2, \quad n = 1, 2, \ldots$
- eigenfunctions: $\sin(\pi n x)$
- effect of initial conditions: $\int_0^1 \sin(\pi n x) u_0(x) dx$

The largest eigenvalue $\lambda - \pi^2$ ($n = 1$) indicates the rate of growth or decay of the solution. We can see that the plant is stable for $\lambda \leq \pi^2$ and is unstable otherwise.

After the transient response due to the initial conditions, the profile of the state will be proportional to the first eigenfunction $\sin(\pi x)$, since other modes decay much faster.

Sometimes it is possible to use the method of separation of variables to determine the stability properties of the plant even though the complete closed form solution cannot be obtained.

Example 2 Let us find the values of the parameter g for which the system

$$u_t = u_{xx} + gu(0) \tag{70}$$

$$u_x(0) = 0 \tag{71}$$

$$u(1) = 0 \tag{72}$$

is unstable.

This example is motivated by the model of thermal instability in solid propellant rockets, where the term $gu(0)$ is roughly the burning of the propellant at one end of the fuel chamber.

Using the method of separation of variables we set $u(x,t) = X(x)T(t)$ and (70) gives:

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + gX(0)}{X(x)} = \sigma\,. \tag{73}$$

Hence, $T(t) = T(0)e^{\sigma t}$, whereas the solution of the ODE for X is given by

$$X(x) = A\sinh(\sqrt{\sigma}x) + B\cosh(\sqrt{\sigma}x) + \frac{g}{\sigma}X(0)\,. \tag{74}$$

Here the last term is a particular solution of a nonhomogeneous ODE (73).

Now we find the constant B in terms of $X(0)$ by setting $x = 0$ in the above equation. This gives $B = X(0)(1 - g/\sigma)$.

Using the boundary condition (71) we get $A = 0$ so that

$$X(x) = X(0)\left[\frac{g}{\sigma} + \left(1 - \frac{g}{\sigma}\right)\cosh(\sqrt{\sigma}x)\right]\,. \tag{75}$$

Using the other boundary condition (72), we get the eigenvalue relationship

$$\frac{g}{\sigma} = \left(\frac{g}{\sigma} - 1\right) \cosh(\sqrt{\sigma})\,. \tag{76}$$

The above equation has no closed form solution. However, in this particular example we can still find the stability region by finding values of g for which there are eigenvalues with zero real parts. First we check if $\sigma = 0$ satisfies (76) for some values of g.

Using the Taylor expansion for $\cosh(\sqrt{\sigma})$, we get

$$\frac{g}{\sigma} = \left(\frac{g}{\sigma} - 1\right)\left(1 + \frac{\sigma}{2} + O(\sigma^2)\right) = \frac{g}{\sigma} - 1 + \frac{g}{2} - \frac{\sigma}{2} + O(\sigma), \tag{77}$$

which gives

$$\boxed{g \to 2 \quad \text{as} \quad \sigma \to 0}$$

To show that there are no other eigenvalues on the imaginary axis, we set $\sigma = 2jy^2$, $y > 0$. Equation (76) then becomes

$$\cosh((j+1)y) = \frac{g}{g - 2jy^2}$$

$$\cos(y)\cosh(y) + j\sin(y)\sinh(y) = \frac{g^2 + 2jgy^2}{g^2 + 4y^4}\,.$$

Taking the absolute value, we get

$$\sinh(y)^2 + \cos(y)^2 = \frac{g^4 + 4g^2y^4}{(g^2 + 4y^4)^2}\,. \tag{78}$$

The only solution to this equation is $y = 0$, which can be seen by computing derivatives of both sides of (78):

$$\frac{d}{dy}(\sinh(y)^2 + \cos(y)^2) = \sinh(2y) - \sin(2y) > 0 \quad \text{for all } y > 0 \tag{79}$$

$$\frac{d}{dy}\frac{g^4 + 4g^2y^4}{(g^2 + 4y^4)^2} = -\frac{16g^2y^3}{(g^2 + 4y^4)^2} < 0 \quad \text{for all } y > 0\,. \tag{80}$$

Therefore, both sides of (78) start at the same point at $y = 0$ and for $y > 0$ the left hand side monotonically grows while the right hand side monotonically decays.

We thus proved that the plant (70)–(72) is neutrally stable only when $g = 2$.

SUMMARY: the plant is stable for $g < 2$ and unstable for $g > 2$.

Notes and References

The method of separation of variables is discussed in detail in classical PDE texts

R. Courant and D. Hilbert, *Methods of mathematical physics*, New York, Interscience Publishers, 1962.

E. Zauderer, *Partial differential equations of applied mathematics*, New York : Wiley, 2nd ed., 1998.

The exact solutions for many problems can be found in

H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Oxford, Clarendon Press, 1959.

A. D. Polianin, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Boca Raton, Fla, Chapman and Hall/CRC, 2002.

Transform methods for PDEs are studied extensively in

D. G. DUFFY, *Transform methods for solving partial differential equations*, Boca Raton, FL : CRC Press, 1994.

Homework

1. Consider the Reaction-Diffusion equation

$$u_t = u_{xx} + \lambda u$$

for $x \in (0,1)$ with the initial condition $u_0(x) = u(x,0)$ and boundary conditions

$$u_x(0) = 0$$
$$u(1) = 0\,.$$

1) Find the solution of this PDE.

2) For what values of the parameter λ is this system unstable?

2. Consider the heat equation

$$u_t = u_{xx}$$

with Robin's boundary conditions

$$u_x(0) = -qu(0)$$
$$u(1) = 0\,.$$

Find the range of values of the parameter q for which this system is unstable.

Backstepping for Parabolic PDEs

(Reaction-Advection-Diffusion and Other Equations)

The most important part of this course.

We introduce the method of *backstepping*, using the class of parabolic PDEs.

Later we extend backstepping to 1st and 2nd-order hyperbolic PDEs and to other classes.

Parabolic PDEs are first order in time and, while they can have a large number of unstable eigenvalues, this number is finite, which makes them more easily accessible to a reader with background in ODEs.

Backstepping is capable of eliminating destabilizing forces/terms acting in the domain's interior, using control that acts only on the boundary.

We build a state transfornation, which involves a Volterra integral operator that 'absorbs' the destabilizing terms acting in the domain and brings them to the boundary, where control can eliminate them.

The Volterra operator has a lower triangular structure.

ODE Backstepping

The *backstepping* method and its name originated in the early 1990's for stabilization of nonlinear ODE systems

M. KRSTIC, I. KANELLAKOPOULOS, AND P. KOKOTOVIC, *Nonlinear and Adaptive Control Design*, Wiley, New York, 1995.

Consider the following three-state nonlinear system

$$\dot{y}_1 = y_2 + y_1^3 \tag{81}$$
$$\dot{y}_2 = y_3 + y_2^3 \tag{82}$$
$$\dot{y}_3 = u + y_3^3 \tag{83}$$

Since the control input u is only in the last equation (83), we view it as boundary control.

The nonlinear terms y_1^3, y_2^3, y_3^3 can be viewed as nonlinear "reaction" terms. They are destabilizing because, for $u = 0$, the overall system is a "cascade" of three unstable subsystems of the form $\dot{y}_i = y_i^3$ (the open-loop system exhibits a finite-time escape instability).

The control u can cancel the "matched" term y_3^3 in (83) but cannot cancel directly the unmatched terms y_1^3 and y_2^3 in (81), (83).

To achieve the cancellation of all three of the destabilizing y_i^3-terms, a backstepping change of variable is constructed recursively,

$$
\begin{aligned}
z_1 &= y_1 && (84)\\
z_2 &= y_2 + y_1^3 + c y_1 && (85)\\
z_3 &= y_3 + y_2^3 + (3y_1^2 + 2c) y_2 + 3y_1^5 + 2c y_1^3 + (c^2+1) y_1\,, && (86)
\end{aligned}
$$

along with the control law

$$
\begin{aligned}
u &= -c_3 z_3 - z_2 - y_3^3 - (3y_2^2 + 3y_1^2 + 2c)(y_3 + y_2^3) \\
&\quad - (6y_1 y_2 + 15 y_1^4 + 6c y_1^2 + c^2 + 1)(y_2 + y_1^3)\,, && (87)
\end{aligned}
$$

which convert the system (81)–(83) into

$$
\begin{aligned}
\dot z_1 &= z_2 - c z_1 && (88)\\
\dot z_2 &= -z_1 + z_3 - c z_2 && (89)\\
\dot z_3 &= -z_2 - c z_3\,, && (90)
\end{aligned}
$$

where the control parameter c should be chosen positive.

The system (88)–(90), which can also be written as

$$\dot{z} = Az \tag{91}$$

where

$$A = \begin{bmatrix} -c & 1 & 0 \\ -1 & -c & 1 \\ 0 & -1 & -c \end{bmatrix}, \tag{92}$$

is exponentially stable because

$$A + A^T = -cI\,. \tag{93}$$

The equality (93) guarantees that the Lyapunov function

$$V = \frac{1}{2} z^T z \tag{94}$$

has a negative definite time derivative

$$\dot{V} = -cz^T z = -2cV\,. \tag{95}$$

Hence, the target system (88)–(90) is desirable.

Let us first examine the change of variables $y \mapsto z$ in (84)–(86):

$$\begin{aligned} z_1 &= \boxed{y_1} \\ z_2 &= \boxed{y_2} + y_1^3 + cy_1 \\ z_3 &= \boxed{y_3} + y_2^3 + (3y_1^2 + 2c)y_2 + 3y_1^5 + 2cy_1^3 + (c^2+1)y_1 \end{aligned}$$

This change of variables is of the form

$$z = (I - K)[y], \tag{96}$$

where I is the identity matrix and K is a "lower-triangular" nonlinear transformation.

The lower-triangular structure of K is a finite-dimensional analog of the spatially-causal Volterra integral operator

$$K[u](x) \triangleq \int_0^{\boxed{x}} k(x,y)u(y)\,dy \tag{97}$$

Hence, for PDEs, in analogy with (96), we use the change of variable

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy. \tag{98}$$

An important feature of the change of variable

$$
\begin{aligned}
z_1 &= \boxed{y_1} \\
z_2 &= \boxed{y_2} + y_1^3 + c y_1 \\
z_3 &= \boxed{y_3} + y_2^3 + (3y_1^2 + 2c) y_2 + 3y_1^5 + 2c y_1^3 + (c^2 + 1) y_1
\end{aligned}
$$

is that it is invertible, i.e., y can be expressed as a smooth function of z (to be specific,

$$
\begin{aligned}
y_1 &= z_1 \\
y_2 &= z_2 - z_1^3 - c z_1 \\
y_3 &= z_3 - \cdots
\end{aligned}
$$

In the PDE case, the transformation $u \mapsto w$ in (98) is also invertible and the inverse can be written as

$$
u(x) = w(x) + \int_0^x l(x,y) w(y)\, dy. \tag{99}
$$

Next, let us examine the relation between the target systems

$$\dot{z} = Az\,, \qquad A = \begin{bmatrix} -c & 1 & 0 \\ -1 & -c & 1 \\ 0 & -1 & -c \end{bmatrix} \tag{100}$$

and the heat equation as a target system for parabolic PDEs,

$$w_t = w_{xx}\,. \tag{101}$$

They both admit a simple 2-norm as a Lyapunov function, specifically,

$$\frac{d}{dt}\frac{1}{2}z^T z = -cz^T z \tag{102}$$

in the ODE case and

$$\frac{d}{dt}\frac{1}{2}\int_0^1 w(x)^2\,dx = -\int_0^1 w_x(x)^2\,dx \tag{103}$$

in the PDE case.

Backstepping for PDEs—the Main Idea

Start with one of the simplest unstable PDEs, the reaction-diffusion equation:

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) + \boxed{\lambda u(x,t)} && (104)\\
u(0,t) &= 0 && (105)\\
u(1,t) &= U(t) = \text{control} && (106)
\end{aligned}
$$

The open-loop system (104), (105) (with $u(1,t) = 0$) is unstable with arbitrarily many unstable eigenvalues for sufficiently large $\lambda > 0$.

Since the term λu is the source of instability, the natural objective for a boundary feedback is to “eliminate” this term.

State transformation

$$\boxed{w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy} \tag{107}$$

Feedback control

$$u(1,t) = \int_0^1 k(1,y)u(y,t)\,dy \tag{108}$$

Target system (exp. stable)

$$\begin{aligned} w_t(x,t) &= w_{xx}(x,t) && (109)\\ w(0,t) &= 0 && (110)\\ w(1,t) &= 0 && (111) \end{aligned}$$

Task: find kernel $k(x,y)$.

The Volterra integral transformation in (107) has the following features:

The limits of integral are from 0 to x, not from 0 to 1.

“Spatially causal,” that is, for a given x the right hand side of (107) depends only on the values of u in the interval $[0,x]$.

Invertible because of the presence of the identity operator and the spatial causality of the Volterra operator. Because of invertibility, stability of the target system translates into stability of the closed loop system consisting of the plant plus boundary feedback.

Gain Kernel PDE

Task: find the function $k(x,y)$ (which we call "gain kernel") that makes the plant (104), (105) with the controller (108) equivalent to the target system (109)–(111).

We introduce the following notation:

$$
\begin{aligned}
k_x(x,x) &= \frac{\partial}{\partial x} k(x,y)|_{y=x} \\
k_y(x,x) &= \frac{\partial}{\partial y} k(x,y)|_{y=x} \\
\frac{d}{dx} k(x,x) &= k_x(x,x) + k_y(x,x).
\end{aligned}
$$

Differentiate the transformation (107) with respect to x and t using Leibnitz's rule

$$\frac{d}{dx}\int_0^x f(x,y)\,dy = f(x,x) + \int_0^x f_x(x,y)\,dy\,.$$

Differentiating the transformation (107) with respect to x gives

$$\begin{aligned}
w_x(x) &= u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy \\
w_{xx}(x) &= u_{xx}(x) - \frac{d}{dx}(k(x,x)u(x)) - k_x(x,x)u(x) - \int_0^x k_{xx}(x,y)u(y)\,dy \\
&= u_{xx}(x) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x) - k_x(x,x)u(x) \\
&\quad - \int_0^x k_{xx}(x,y)u(y)\,dy\,. \qquad (112)
\end{aligned}$$

Next, we differentiate the transformation (107) with respect to time:

$$
\begin{aligned}
w_t(x) &= u_t(x) - \int_0^x k(x,y)u_t(y)dy \\
&= u_{xx}(x) + \lambda u(x) - \int_0^x k(x,y)\left(u_{yy}(y) + \lambda u(y)\right) dy \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) \\
&\quad + \int_0^x k_y(x,y)u_y(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy \quad \text{(integration by parts)} \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy\,. \quad \text{(integration by parts)}
\end{aligned}
\tag{113}
$$

Subtracting (112) from (113), we get

$$
\boxed{
\begin{aligned}
w_t - w_{xx} &= \left[\lambda + 2\frac{d}{dx}k(x,x)\right] u(x) + k(x,0)u_x(0) \\
&\quad + \int_0^x \left(k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)\right) u(y)\,dy \\
&= 0
\end{aligned}
}
$$

For this to hold for all u, three conditions have to be satisfied:

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y) &= 0 && (114)\\
k(x,0) &= 0 && (115)\\
\lambda + 2\frac{d}{dx}k(x,x) &= 0. && (116)
\end{aligned}
$$

We simplify (116) by integrating it with respect to x and noting from (115) that $k(0,0) = 0$, which gives us

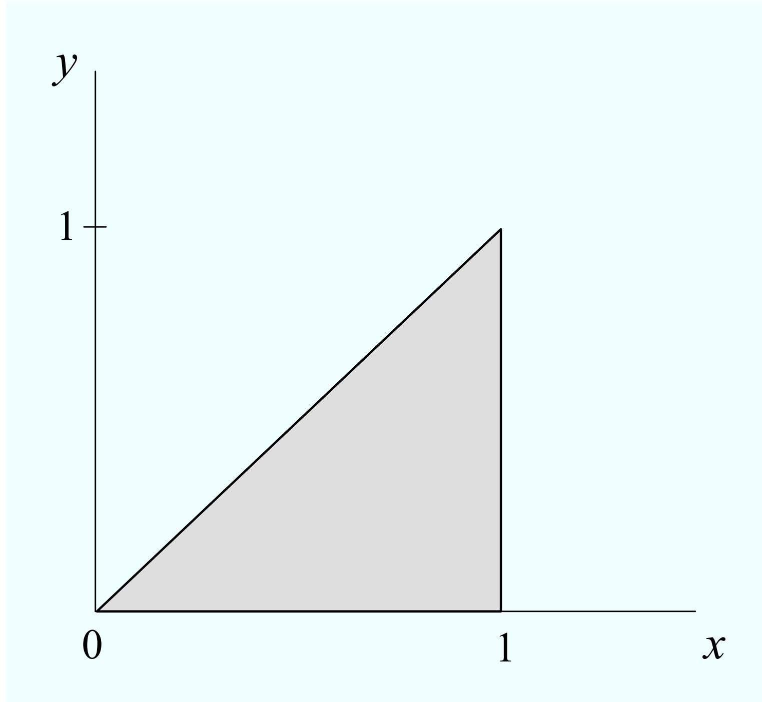
$$
\boxed{
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y)\\
k(x,0) &= 0\\
k(x,x) &= -\frac{\lambda}{2}x
\end{aligned}
}
\tag{117}
$$

These three conditions form a well posed PDE of hyperbolic type in the “Goursat form.”

One can think of the k-PDE as a wave equation with an extra term λk.

x plays the role of time and y of space.

In quantum physics such PDEs are called Klein-Gordon PDEs.



Domain of the PDE for gain kernel $k(x,y)$.

The boundary conditions are prescribed on hypotenuse and the lower cathetus of the triangle.

The value of $k(x,y)$ on the vertical cathetus gives us the control gain $k(1,y)$.

Converting Gain Kernel PDE to an Integral Equation

To find a solution of the k-PDE (117) we first convert it into an integral equation.

Introducing the change of variables

$$\boxed{\xi = x+y, \qquad \eta = x-y} \tag{118}$$

we have

$$\begin{aligned} k(x,y) &= G(\xi,\eta) \\ k_x &= G_\xi + G_\eta \\ k_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} \\ k_y &= G_\xi - G_\eta \\ k_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}\,. \end{aligned}$$

Thus, the gain kernel PDE becomes

$$G_{\xi\eta}(\xi,\eta) = \frac{\lambda}{4}G(\xi,\eta) \tag{119}$$
$$G(\xi,\xi) = 0 \tag{120}$$
$$G(\xi,0) = -\frac{\lambda}{4}\xi. \tag{121}$$

Integrating (119) with respect to η from 0 to η, we get

$$G_\xi(\xi,\eta) = G_\xi(\xi,0) + \int_0^\eta \frac{\lambda}{4}G(\xi,s)\,ds = -\frac{\lambda}{4} + \int_0^\eta \frac{\lambda}{4}G(\xi,s)\,ds. \tag{122}$$

Next, we integrate (122) with respect to ξ from η to ξ to get the <u>integral equation</u>

$$\boxed{G(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G(\tau,s)\,ds\,d\tau} \tag{123}$$

The G-integral eqn is easier to analyze than the k-PDE.

Method of Successive Approximations

Start with an initial guess

$$G^0(\xi,\eta) = 0 \tag{124}$$

and set up the recursive formula for (123) as follows:

$$\boxed{G^{n+1}(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_{\eta}^{\xi}\int_{0}^{\eta} G^n(\tau,s)\,ds\,d\tau} \tag{125}$$

If this functional iteration converges, we can write the solution $G(\xi,\eta)$ as

$$G(\xi,\eta) = \lim_{n\to\infty} G^n(\xi,\eta)\,. \tag{126}$$

Let us denote the difference between two consecutive terms as

$$\Delta G^n(\xi,\eta) = G^{n+1}(\xi,\eta) - G^n(\xi,\eta)\,. \tag{127}$$

Then

$$\Delta G^{n+1}(\xi,\eta) = \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta \Delta G^n(\tau,s)\,ds\,d\tau \tag{128}$$

and (126) can be alternatively written as

$$G(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G^n(\xi,\eta)\,. \tag{129}$$

Computing ΔG^n from (128) starting with

$$\Delta G^0 = G^1(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta)\,, \tag{130}$$

we can observe the pattern which leads to the following formula:

$$\boxed{\Delta G^n(\xi,\eta) = -\frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{\lambda}{4}\right)^{n+1}} \tag{131}$$

This formula can be verified by induction.

The solution to the integral equation is given by

$$G(\xi,\eta) = -\sum_{n=0}^{\infty} \frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{\lambda}{4}\right)^{n+1} . \tag{132}$$

To compute the series (132), note that a first order modified Bessel function of the first kind can be represented as

$$I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!} . \tag{133}$$

ASIDE: Modified Bessel Functions I_n

The function $y(x) = I_n(x)$ is a solution to the following ODE

$$x^2 y'' + xy' - (x^2 + n^2)y = 0 \tag{134}$$

Series representation

$$\boxed{I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!}} \tag{135}$$

Properties

$$2nI_n(x) = x(I_{n-1}(x) - I_{n+1}(x)) \tag{136}$$

$$I_n(-x) = (-1)^n I_n(x) \tag{137}$$

Differentiation

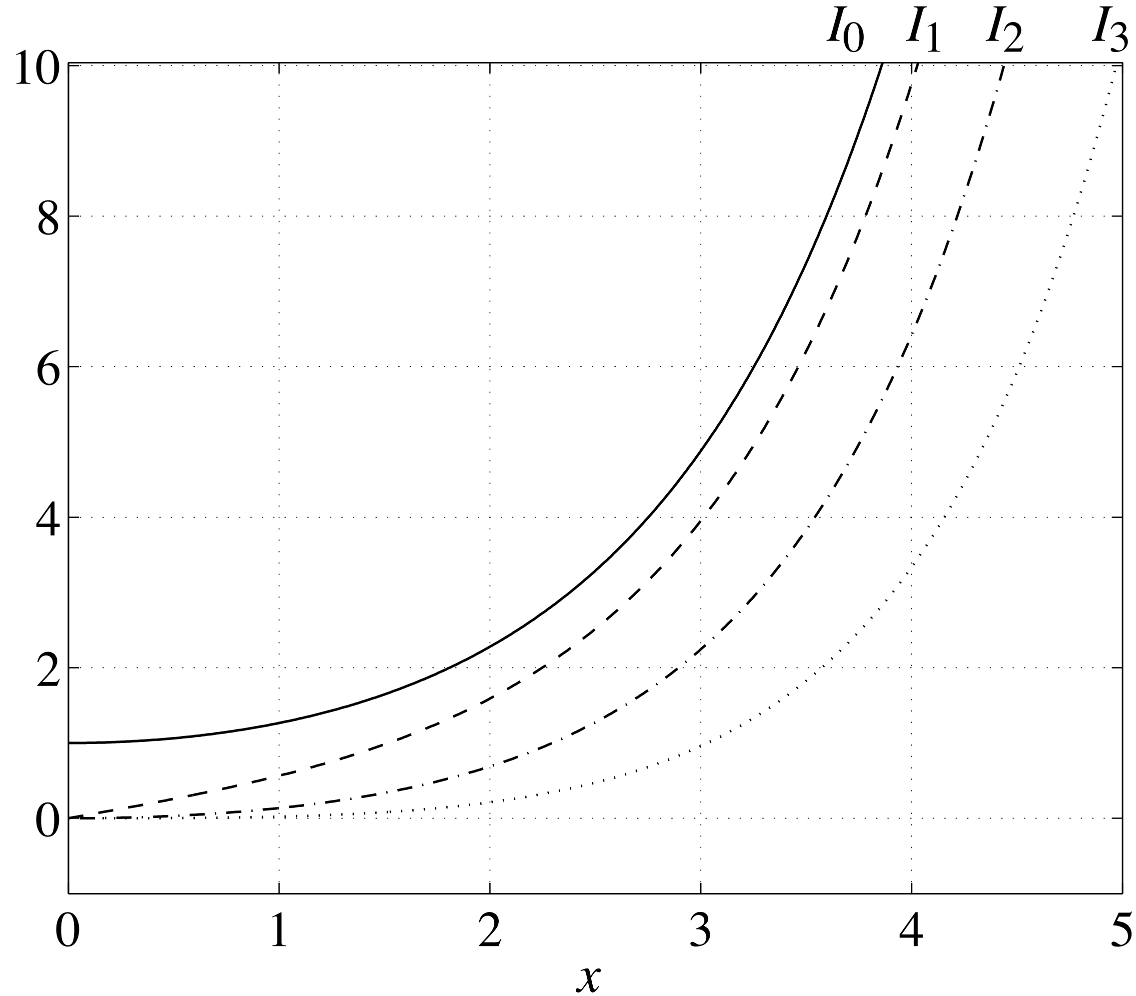
$$\frac{d}{dx}I_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) = \frac{n}{x}I_n(x) + I_{n+1}(x) \tag{138}$$

$$\frac{d}{dx}(x^n I_n(x)) = x^n I_{n-1}, \qquad \frac{d}{dx}(x^{-n} I_n(x)) = x^{-n} I_{n+1} \tag{139}$$

Asymptotic properties

$$I_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \to 0 \tag{140}$$

$$I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \qquad x \to \infty \tag{141}$$



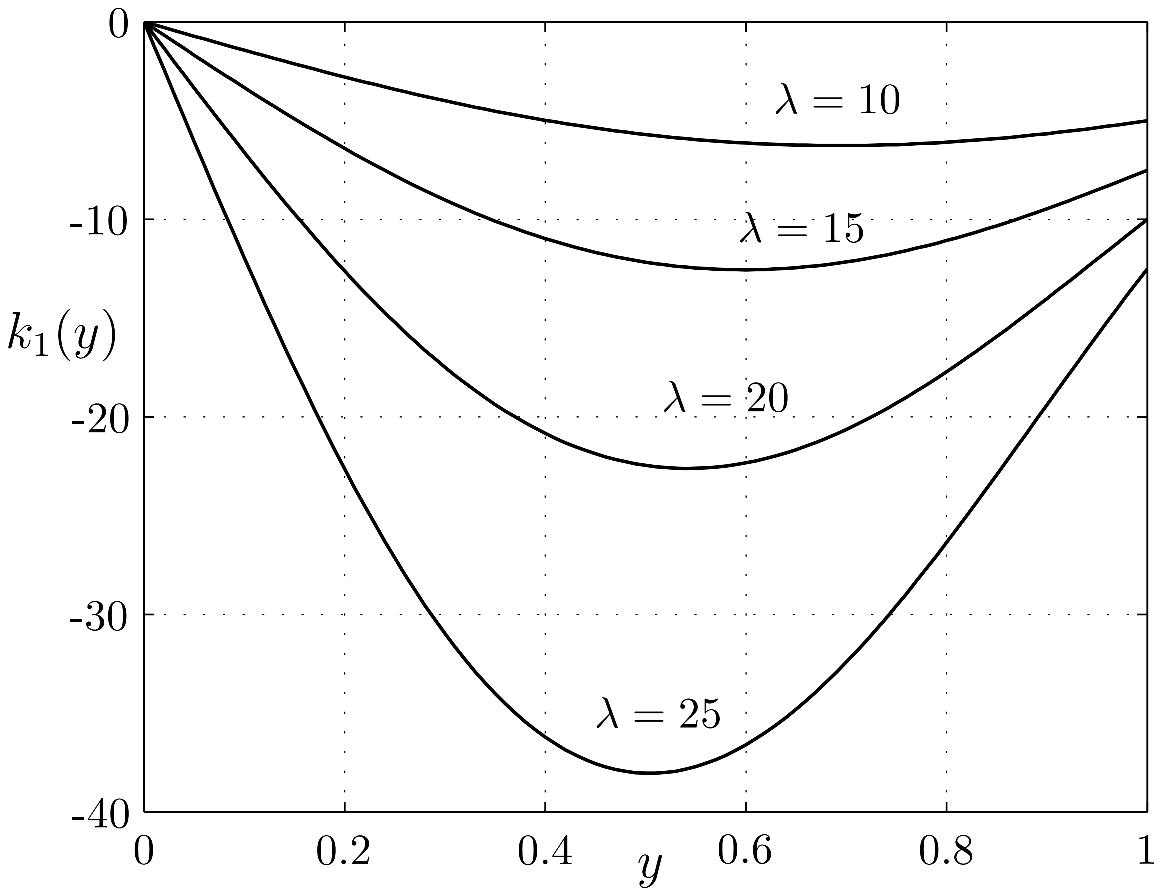
Modified Bessel functions I_n.

Comparing (135) with (132) we obtain

$$G(\xi,\eta) = -\frac{\lambda}{2}(\xi-\eta)\frac{I_1(\sqrt{\lambda\xi\eta})}{\sqrt{\lambda\xi\eta}} \tag{142}$$

or, returning to the original x, y variables,

$$\boxed{k(x,y) = -\lambda y\frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}} \tag{143}$$



Control gain $k(1, y)$ for different values of λ

As λ gets larger, the plant becomes more unstable which requires more control effort.

Low gain near the boundaries is logical: near $x = 0$ the state is small even without control because of the boundary condition $u(0) = 0$; near $x = 1$ the control has the most impact.

Inverse Transformation

We need to establish that stability of the w-target system (109)–(111) implies stability of the u-closed-loop system (104), (105), (108), by showing that the transformation $u \mapsto w$ is invertible.

Postulate an inverse transformation in the form

$$u(x) = w(x) + \int_0^x l(x,y)w(y)\,dy\,, \tag{144}$$

where $l(x,y)$ is the transformation kernel.

Given the direct transformation (107) and the inverse transformation (144), the kernels $k(x,y)$ and $l(x,y)$ satisfy

$$\boxed{l(x,y) = k(x,y) + \int_y^x k(x,\xi)l(\xi,y)\,d\xi} \tag{145}$$

Proof of (145)**.** First recall from calculus the following formula for changing the order of integration:

$$\boxed{\int_0^x \int_0^y f(x,y,\xi)\, d\xi\, dy = \int_0^x \int_\xi^x f(x,y,\xi)\, dy\, d\xi} \tag{146}$$

Substituting (144) into (107), we get

$$\begin{aligned} w(x) &= w(x) + \int_0^x l(x,y)w(y)dy - \int_0^x k(x,y)\left[w(y) + \int_0^y l(y,\xi)w(\xi)d\xi\right] dy \\ &= w(x) + \int_0^x l(x,y)w(y)dy - \int_0^x k(x,y)w(y)dy - \int_0^x \int_0^y k(x,y)l(y,\xi)w(\xi)d\xi dy \\ 0 &= \int_0^x w(y)\left[l(x,y) - k(x,y) - \int_y^x k(x,\xi)l(\xi,y)\, d\xi\right] dy\,. \end{aligned}$$

Since the last line has to hold for all $w(y)$, we get the relationship (145). ■

The formula (145) is general (it does not depend on the plant and the target system) but is not very helpful in actually finding $l(x,y)$ from $k(x,y)$.

Instead, we follow the same approach that led us to the kernel PDE for $k(x,y)$.

Differentiating (144) with respect to time we get

$$\begin{aligned} u_t(x) &= w_t(x) + \int_0^x l(x,y) w_t(y)\, dy \\ &= w_{xx}(x) + l(x,x) w_x(x) - l(x,0) w_x(0) - l_y(x,x) w(x) \\ &\quad + \int_0^x l_{yy}(x,y) w(y)\, dy \end{aligned} \tag{147}$$

and differentiating twice with respect to x gives

$$\begin{aligned} u_{xx}(x) &= w_{xx}(x) + l_x(x,x) w(x) + w(x) \frac{d}{dx} l(x,x) + l(x,x) w_x(x) \\ &\quad + \int_0^x l_{xx}(x,y) w(y)\, dy\,. \end{aligned} \tag{148}$$

Subtracting (148) from (147) we get

$$\begin{aligned}\lambda w(x)+\lambda\int_0^x l(x,y)w(y)dy &= -2w(x)\frac{d}{dx}l(x,x)-l(x,0)w_x(0)\\ &\quad+\int_0^x (l_{yy}(x,y)-l_{xx}(x,y))w(y)\,dy\end{aligned}$$

which gives the following conditions on $l(x,y)$:

$$\boxed{\begin{aligned}l_{xx}(x,y)-l_{yy}(x,y) &= -\lambda l(x,y)\\ l(x,0) &= 0\\ l(x,x) &= -\frac{\lambda}{2}x\end{aligned}} \tag{149}$$

Comparing this PDE with the PDE (117) for $k(x,y)$, we see that

$$l(x,y;\lambda) = -k(x,y;-\lambda)\,. \tag{150}$$

From (143) we have

$$l(x,y) = -\lambda y \frac{I_1\left(\sqrt{-\lambda(x^2-y^2)}\right)}{\sqrt{-\lambda(x^2-y^2)}} = -\lambda y \frac{I_1\left(j\sqrt{\lambda(x^2-y^2)}\right)}{j\sqrt{\lambda(x^2-y^2)}},$$

or, using the properties of I_1,

$$\boxed{l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}} \tag{151}$$

ASIDE: Bessel Functions J_n

The function $y(x) = J_n(x)$ is a solution to the following ODE

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \tag{152}$$

Series representation

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!} \tag{153}$$

Relationship with $J_n(x)$

$$I_n(x) = i^{-n} J_n(ix), \qquad I_n(ix) = i^n J_n(x) \tag{154}$$

Properties

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x)) \tag{155}$$

$$J_n(-x) = (-1)^n J_n(x) \tag{156}$$

Differentiation

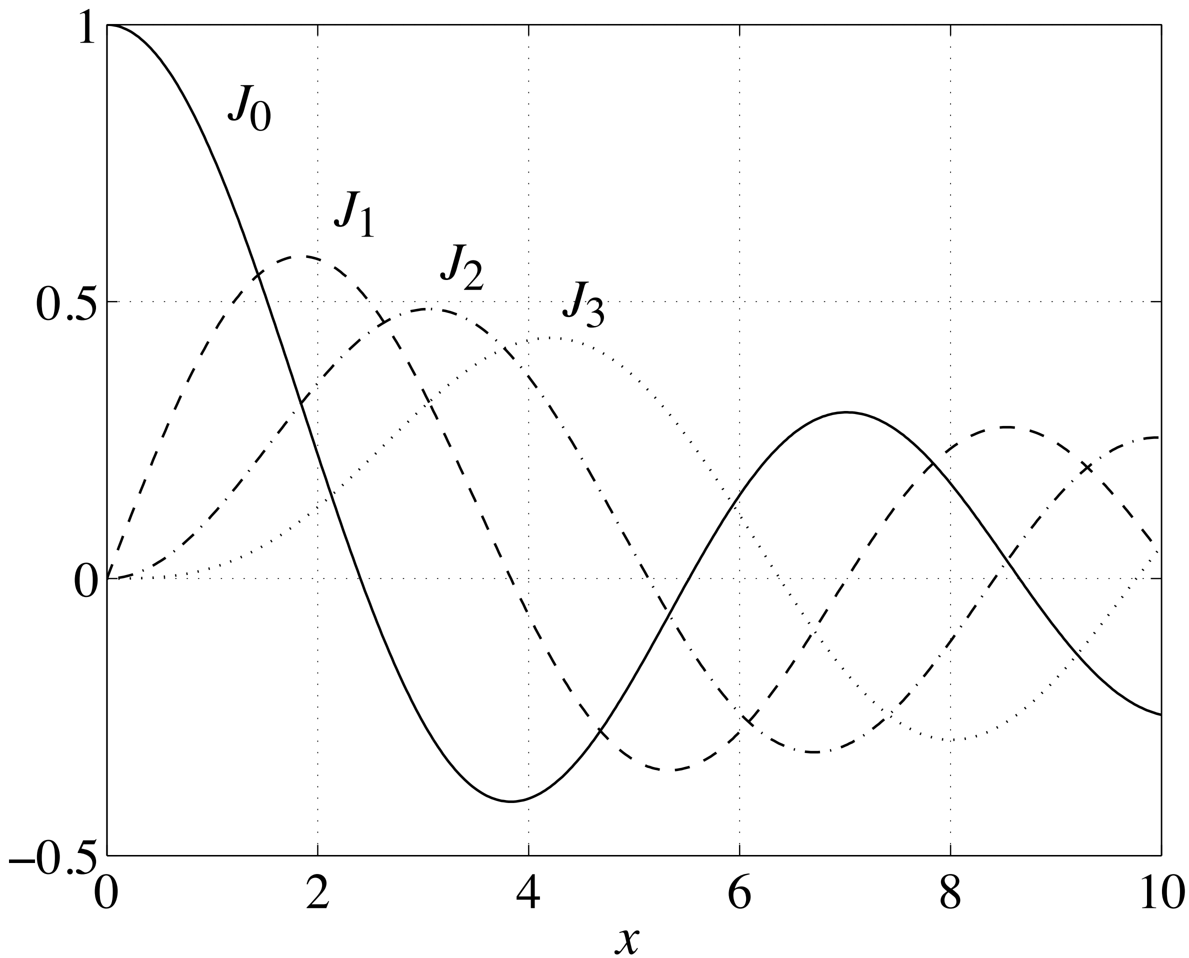
$$\frac{d}{dx}J_n(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x)) = \frac{n}{x}J_n(x) - J_{n+1}(x) \tag{157}$$

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}, \qquad \frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1} \tag{158}$$

Asymptotic properties

$$J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad x \longrightarrow 0 \tag{159}$$

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi n}{2} - \frac{\pi}{4}\right), \qquad x \longrightarrow \infty \tag{160}$$



Bessel functions J_n.

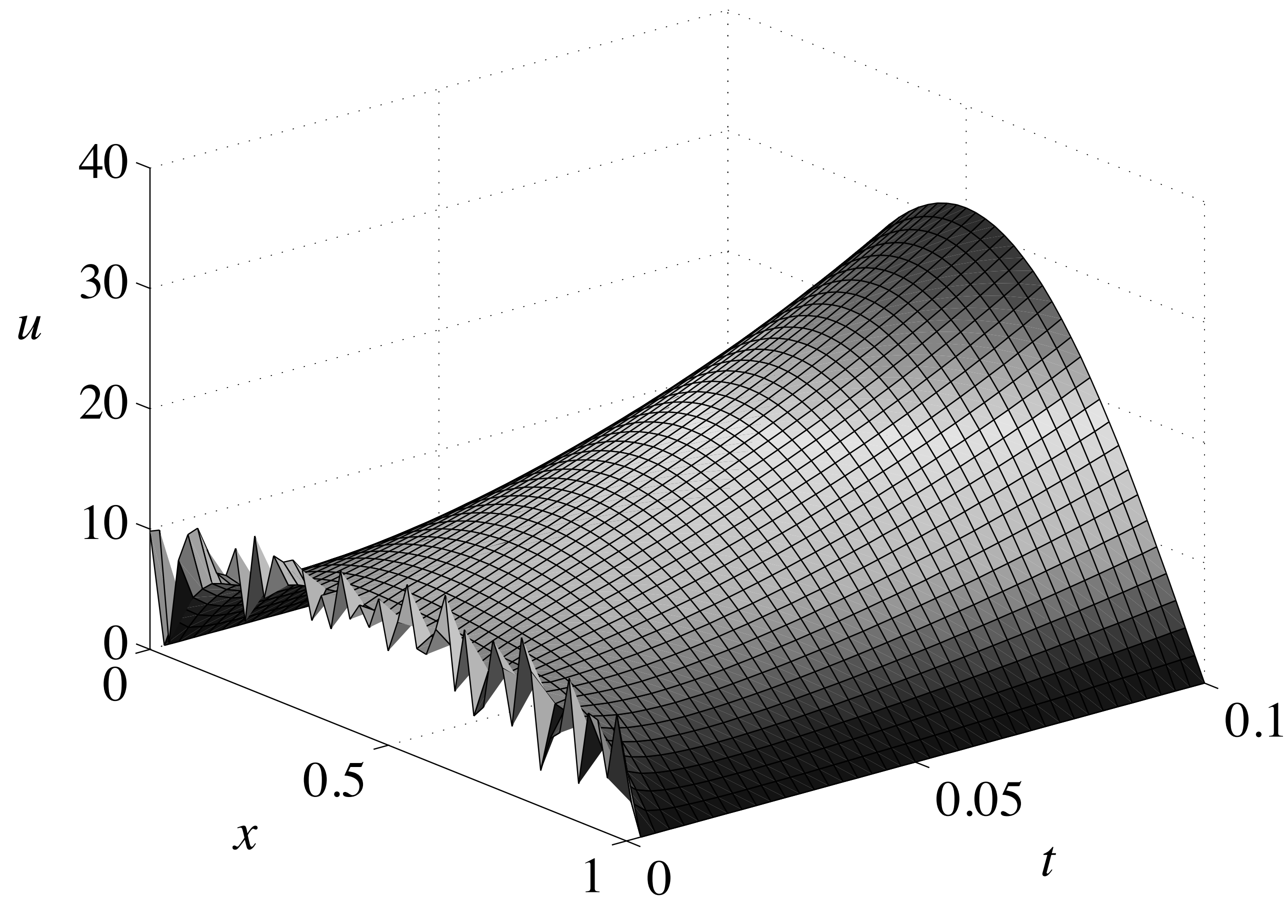
Summary of control design for the reaction-diffusion equation

Plant
$$u_t = u_{xx} + \lambda u \tag{161}$$
$$u(0) = 0 \tag{162}$$

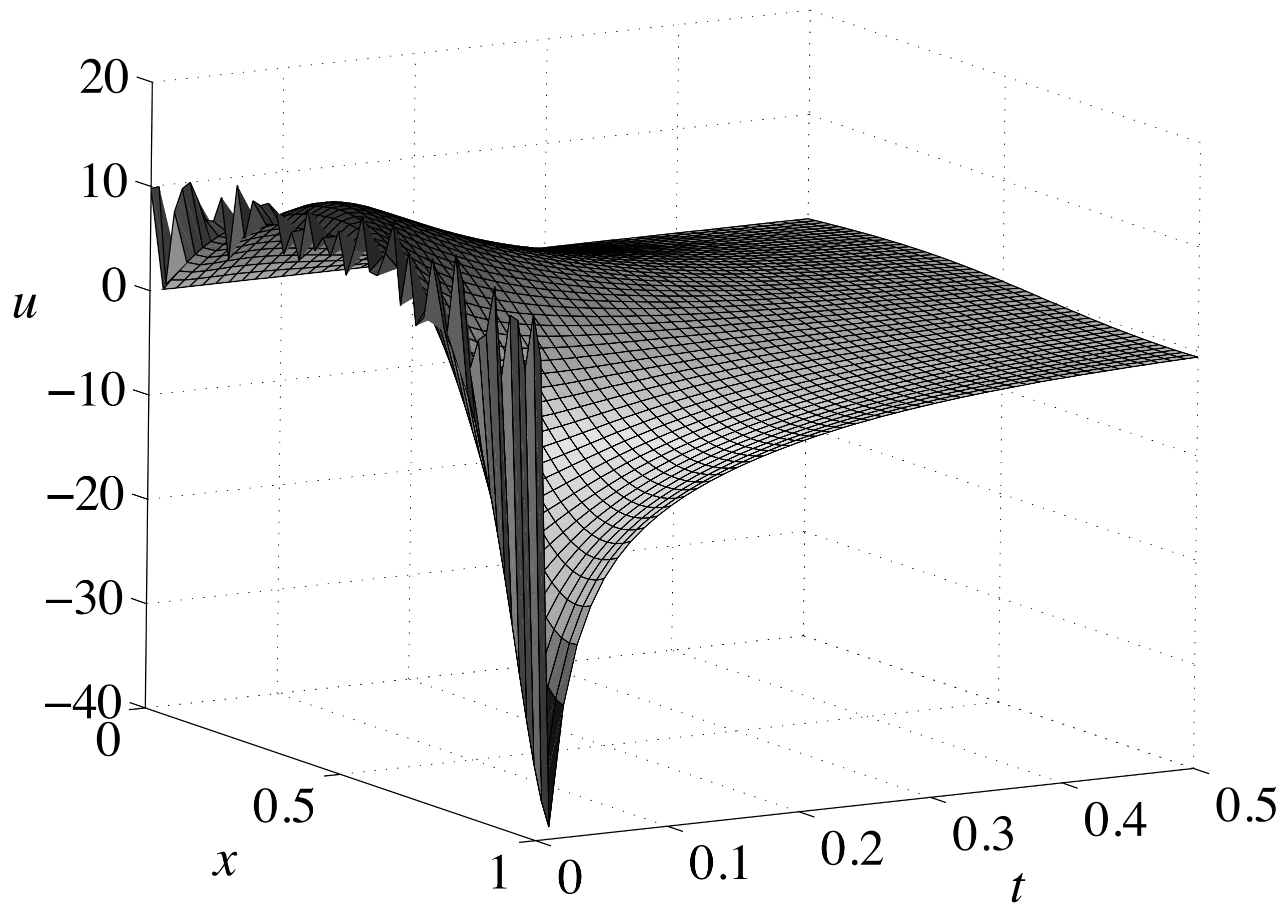
Controller
$$u(1) = -\int_0^1 y\lambda \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} u(y)\,dy \tag{163}$$

Transformation
$$w(x) = u(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} u(y)\,dy \tag{164}$$
$$u(x) = w(x) - \int_0^x \lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} w(y)\,dy \tag{165}$$

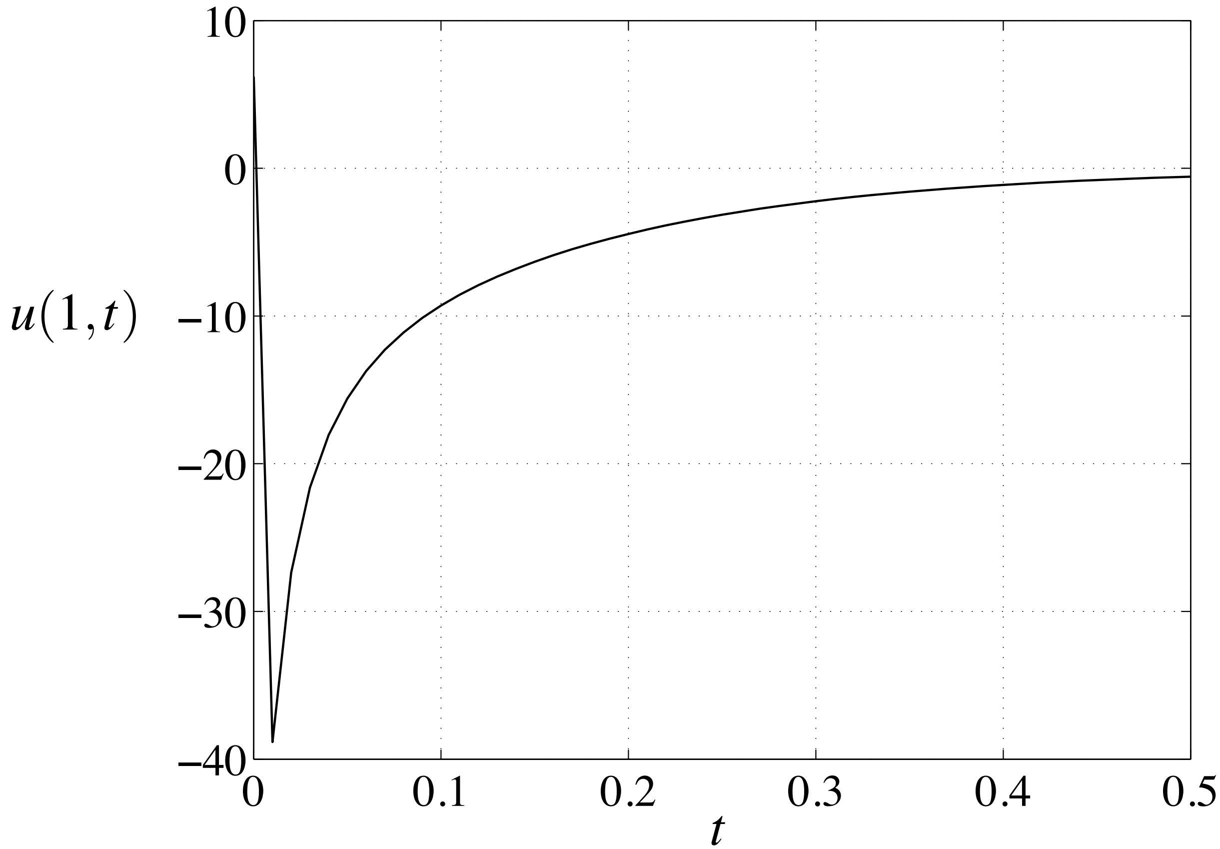
Target system
$$w_t = w_{xx} \tag{166}$$
$$w(0) = 0 \tag{167}$$
$$w(1) = 0 \tag{168}$$



Open-loop response for reaction-diffusion plant (161), (162) for the case $\lambda = 20$.
The plant has one unstable eigenvalue $20 - \pi^2 \approx 10$.



Closed-loop response with controller (163) implemented.



The control (163) for reaction-diffusion plant (161)–(162).

Example 3 Consider the plant with a Neumann boundary cond. on the uncontrolled end,

$$
\begin{aligned}
u_t &= u_{xx} + \lambda u && (169)\\
u_x(0) &= 0 && (170)\\
u(1) &= U(t)\,. && (171)
\end{aligned}
$$

We use the transformation

$$
w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy \qquad (172)
$$

to map this plant into the target system

$$
\begin{aligned}
w_t &= w_{xx} && (173)\\
w_x(0) &= 0 && (174)\\
w(1) &= 0\,. && (175)
\end{aligned}
$$

Differentiation of the transformation (172) with respect to x gives (112) (it does not depend on the particular plant). Differentiating (172) with respect to time, we get

$$\begin{aligned}
w_t(x) &= u_t(x) - \int_0^x k(x,y)u_t(y)\,dy \\
&= u_{xx}(x) + \lambda u(x) - \int_0^x k(x,y)[u_{yy}(y) + \lambda u(y)]\,dy \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) \\
&\quad + \int_0^x k_y(x,y)u_y(y)dy - \int_0^x \lambda k(x,y)u(y)\,dy \quad \text{(integration by parts)} \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy \quad \text{(integration by parts)}
\end{aligned} \tag{176}$$

Subtracting (112) from (176), we get

$$
\begin{aligned}
w_t - w_{xx} &= \left[\lambda + 2\frac{d}{dx}k(x,x)\right] u(x) - k_y(x,0)u(0) \\
&\quad + \int_0^x \big(k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)\big)\, u(y)\, dy\,. \qquad (177)
\end{aligned}
$$

For the right hand side of this equation to be zero for all $u(x)$, three conditions must be satisfied:

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y) &= 0 \qquad (178) \\
k_y(x,0) &= 0 \qquad (179) \\
\lambda + 2\frac{d}{dx}k(x,x) &= 0\,. \qquad (180)
\end{aligned}
$$

Integrating (180) with respect to x gives $k(x,x) = -\lambda/2x + k(0,0)$, where $k(0,0)$ is obtained using the boundary condition (174):

$$
w_x(0) = u_x(0) + k(0,0)u(0) = 0\,,
$$

so that $k(0,0) = 0$. The gain kernel PDE is thus

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y) \qquad (181) \\
k_y(x,0) &= 0 \qquad (182) \\
k(x,x) &= -\frac{\lambda}{2}x\,. \qquad (183)
\end{aligned}
$$

Note that this PDE is very similar to (117). The only difference is in the boundary condition at $y = 0$. The solution to the PDE (181)–(183) is obtained through a summation of successive approximation series, similarly to the way it was obtained for the PDE (117):

$$\boxed{k(x,y) = -\lambda x \frac{I_1\left(\sqrt{\lambda(x^2 - y^2)}\right)}{\sqrt{\lambda(x^2 - y^2)}}} \tag{184}$$

Thus, the controller is given by

$$u(1) = -\int_0^1 \lambda \frac{I_1\left(\sqrt{\lambda(1 - y^2)}\right)}{\sqrt{\lambda(1 - y^2)}} u(y)\, dy\,. \tag{185}$$

Neumann Actuation

Consider the plant (104), (105) but with the heat flux $u_x(1)$ actuated:

$$\begin{aligned} u_t &= u_{xx} + \lambda u && (186)\\ u(0) &= 0 && (187)\\ \boxed{u_x(1)} &= U(t)\,. && (188) \end{aligned}$$

We use the same transformation (107), (143) as we used in the case of Dirichlet actuation. To obtain the control $u_x(1)$, we need to differentiate (107) with respect to x:

$$w_x(x) = u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy$$

and set $x = 1$. It is clear now that the target system has to have the Neumann boundary condition at $x = 1$:

$$\begin{aligned} w_t &= w_{xx} && (189)\\ w(0) &= 0 && (190)\\ w_x(1) &= 0\,, && (191) \end{aligned}$$

which gives the controller

$$u_x(1) = k(1,1)u(1) + \int_0^1 k_x(1,y)u(y)\,dy\,. \qquad (192)$$

All that remains is to derive the expression for k_x from (143) using the properties of Bessel functions:

$$k_x(x,y) \quad = \quad -\lambda y x \frac{I_2\left(\sqrt{\lambda(x^2-y^2)}\right)}{x^2-y^2}.$$

Finally, the controller is

$$u_x(1) = -\frac{\lambda}{2}u(1) - \int_0^1 \lambda y \frac{I_2\left(\sqrt{\lambda(1-y^2)}\right)}{1-y^2} u(y)\,dy. \tag{193}$$

Reaction-Advection-Diffusion Equation

$$u_t = \varepsilon u_{xx} + b u_x + \lambda u \tag{194}$$

$$u(0) = 0 \tag{195}$$

$$u(1) = U(t) \tag{196}$$

First, we eliminate the advection term u_x with the following change of variable:

$$\boxed{v(x) = u(x)\mathrm{e}^{\frac{b}{2\varepsilon}x}} \tag{197}$$

Taking the temporal and spatial derivatives, we get

$$\begin{aligned}
u_t(x) &= v_t(x)\mathrm{e}^{-\frac{b}{2\varepsilon}x} \\
u_x(x) &= v_x(x)\mathrm{e}^{-\frac{b}{2\varepsilon}x} - \frac{b}{2\varepsilon}v(x)\mathrm{e}^{-\frac{b}{2\varepsilon}x} \\
u_{xx}(x) &= v_{xx}(x)e^{-\frac{b}{2\varepsilon}x} - \frac{b}{\varepsilon}v_x(x)\mathrm{e}^{-\frac{b}{2\varepsilon}x} + \frac{b^2}{4\varepsilon^2}v(x)\mathrm{e}^{-\frac{b}{2\varepsilon}x}
\end{aligned}$$

In the v-variable we get a reaction-diffusion system

$$\begin{aligned} v_t &= \varepsilon v_{xx} + \left(\lambda - \frac{b^2}{4\varepsilon}\right) v && (198) \\ v(0) &= 0 && (199) \\ v(1) &= u(1)\mathrm{e}^{\frac{b}{2\varepsilon}} = \text{control.} && (200) \end{aligned}$$

Now the transformation

$$w(x) = v(x) - \int_0^x k(x,y)v(y)dy \qquad (201)$$

leads to the target system

$$\begin{aligned} w_t &= \varepsilon w_{xx} - cw && (202) \\ w(0) &= 0 && (203) \\ w(1) &= 0\,. && (204) \end{aligned}$$

Here the constant c is a design parameter that sets the decay rate of the closed loop system. It should satisfy the following stability condition:

$$c \geq \max\left\{\frac{b^2}{4\varepsilon} - \lambda, 0\right\}.$$

The max is used to prevent spending unnecessary control effort when the plant is stable.

The gain kernel $k(x,y)$ can be shown to satisfy the following PDE:

$$\begin{aligned}
\varepsilon k_{xx}(x,y) - \varepsilon k_{yy}(x,y) &= \left(\lambda - \frac{b^2}{4\varepsilon} + c\right) k(x,y) && (205)\\
k(x,0) &= 0 && (206)\\
k(x,x) &= -\frac{x}{2\varepsilon}\left(\lambda - \frac{b^2}{4\varepsilon} + c\right). && (207)
\end{aligned}$$

This equation is exactly the same as (117), just with a different constant instead of λ,

$$\lambda_0 = \frac{1}{\varepsilon}\left(\lambda - \frac{b^2}{4\varepsilon} + c\right). \tag{208}$$

Therefore the solution to (205)–(207) is given by

$$k(x,y) = -\lambda_0 y \frac{I_1\left(\sqrt{\lambda_0(x^2-y^2)}\right)}{\sqrt{\lambda_0(x^2-y^2)}}. \tag{209}$$

The controller is

$$u(1) = \int_0^1 \mathrm{e}^{-\frac{b}{2\varepsilon}(1-y)} \lambda_0 y \frac{I_1\left(\sqrt{\lambda_0(1-y^2)}\right)}{\sqrt{\lambda_0(1-y^2)}} u(y)\, dy\,. \tag{210}$$

Let us examine the effect of the advection term bu_x in (194) on open-loop stability and on the size of the control gain. From (198) we see that the advection term has a beneficial effect on open-loop stability, irrespective of the sign of the advection coefficient b. However, the effect of b on the gain function in the control law in (210) is 'sign-sensitive.' Negative values of b demand much higher control effort than positive values of b. Interestingly, negative values of b refer to the situation where the state disturbances advect towards the actuator at $x = 1$, whereas the 'easier' case of positive b refers to the case where the state disturbances advect away from the actuator at $x = 1$ and towards the Dirichlet boundary condition (195) at $x = 0$.

Reaction-Advection-Diffusion Systems w/ Spatially Varying Coeffs

$$\begin{aligned} u_t &= \varepsilon(x)u_{xx} + b(x)u_x + \lambda(x)u && (211) \\ u_x(0) &= -qu(0) && (212) \\ u(1) &= \text{control} && (213) \end{aligned}$$

Systems with thermal, fluid, and chemically reacting dynamics.

The spatially varying coefficients come from applications with non-homogenous materials, unusually shaped domains, and can also arise from linearization.

(Also note the mixed boundary condition at $x = 0$.)

Consider a coordinate change

$$z = \sqrt{\varepsilon_0}\int_0^x \frac{ds}{\sqrt{\varepsilon(s)}}, \quad \text{where } \varepsilon_0 = \left(\int_0^1 \frac{ds}{\sqrt{\varepsilon(s)}}\right)^{-2} \tag{214}$$

and a change of the state variable

$$v(z) = \frac{1}{\sqrt[4]{\varepsilon(x)}} \mathrm{e}^{\int_0^x \frac{b(s)}{2\varepsilon(s)} ds} u(x)\,. \tag{215}$$

Then v satisfies the PDE:

$$\begin{aligned} v_t(z,t) &= \varepsilon_0 v_{zz}(z,t) + \lambda_0(z) v(z,t) \quad (216) \\ v_z(0,t) &= -q_0 v(0,t), \quad (217) \end{aligned}$$

where

$$\begin{aligned} \varepsilon_0 &= \left(\int_0^1 \frac{ds}{\sqrt{\varepsilon(s)}} \right)^{-2} \quad (218) \\ \lambda_0(z) &= \lambda(x) + \frac{\varepsilon''(x)}{4} - \frac{b'(x)}{2} - \frac{3}{16}\frac{(\varepsilon'(x))^2}{\varepsilon(x)} + \frac{1}{2}\frac{b(x)\varepsilon'(x)}{\varepsilon(x)} - \frac{1}{4}\frac{b^2(x)}{\varepsilon(x)} \quad (219) \\ q_0 &= q\sqrt{\frac{\varepsilon(0)}{\varepsilon_0}} - \frac{b(0)}{2\sqrt{\varepsilon_0 \varepsilon(0)}} - \frac{\varepsilon'(0)}{4\sqrt{\varepsilon_0 \varepsilon(0)}}. \quad (220) \end{aligned}$$

We use the transformation (107) to map the modified plant into the target system

$$\begin{aligned} w_t &= \varepsilon_0 w_{zz} - cw \quad (221) \\ w_z(0) &= 0 \quad (222) \\ w(1) &= 0\,. \quad (223) \end{aligned}$$

The transformation kernel is found by solving the PDE

$$
\begin{aligned}
k_{zz}(z,y) - k_{yy}(z,y) &= \frac{\lambda_0(y)+c}{\varepsilon_0} k(z,y) && (224)\\
k_y(z,0) &= -q_0 k(z,0) && (225)\\
k(z,z) &= -q_0 - \frac{1}{2\varepsilon_0}\int_0^z (\lambda_0(y)+c)\,dy\,. && (226)
\end{aligned}
$$

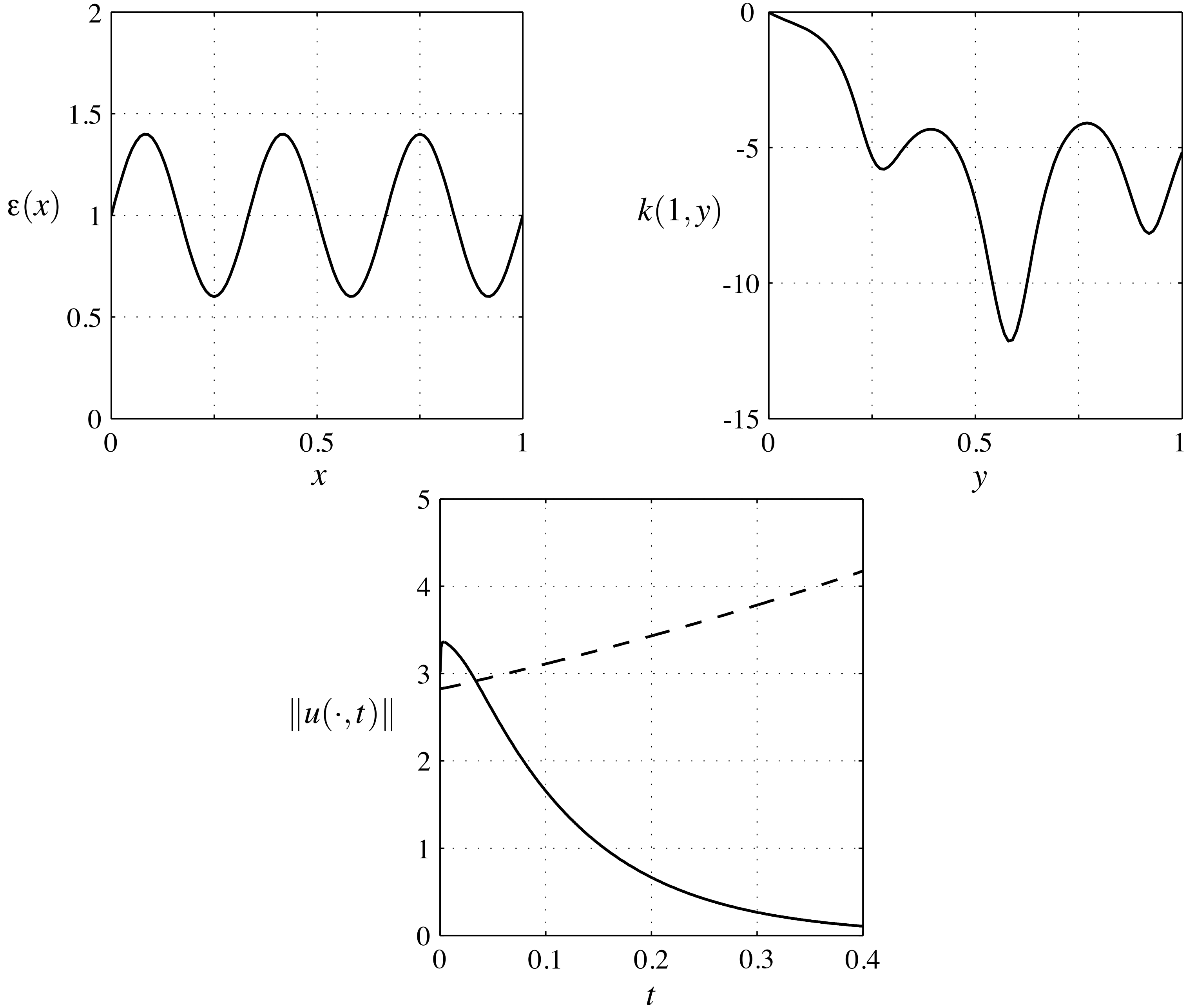
Well posed but cannot be solved in closed form. One can solve it either symbolically, using the successive approximation series, or numerically with finite difference schemes.

Since the controller for v-system is given by

$$
v(1) = \int_0^1 k(1,y)v(y)\,dy\,, \tag{227}
$$

using (214) and (215) we obtain the controller for the original u-plant:

$$
u(1) = \int_0^1 \frac{\varepsilon^{1/4}(1)\sqrt{\varepsilon_0}}{\varepsilon^{3/4}(y)} \mathrm{e}^{-\int_y^1 \frac{b(s)}{2\varepsilon(s)}\,ds} k\left(\int_0^1 \sqrt{\frac{\varepsilon_0}{\varepsilon(s)}}ds, \int_0^y \sqrt{\frac{\varepsilon_0}{\varepsilon(s)}}ds\right) u(y)\,dy\,. \tag{228}
$$



Simulation results for (211)–(212) with controller (228) for $\varepsilon(x) = 1 + 0.4\sin(6\pi x)$, $b \equiv 0$, and $\lambda = 10$.

Other Spatially Causal Plants

$$u_t = u_{xx} + g(x)u(0) + \int_0^x f(x,y)u(y)dy \tag{229}$$

$$u_x(0) = 0\,, \tag{230}$$

where $u(1)$ is actuated.

Equation partly motivated by the model of unstable burning in solid propellant rockets

D. M. BOSKOVIC AND M. KRSTIC, *Stabilization of a solid propellant rocket instability by state feedback*, Int. J. of Robust and Nonlinear Control, vol. 13, pp. 483–495, 2003.

and the thermal convection loop

R. VAZQUEZ AND M. KRSTIC, *Explicit integral operator feedback for local stabilization of nonlinear thermal convection loop PDEs*, Systems and Control Letters, vol. 55, pp. 624–632, 2006.

PDE for the gain kernel:

$$
\begin{aligned}
k_{xx} - k_{yy} &= -f(x,y) + \int_y^x k(x,\xi) f(\xi,y)\, d\xi & (231)\\
k_y(x,0) &= g(x) - \int_0^x k(x,y) g(y)\, dy & (232)\\
k(x,x) &= 0\,. & (233)
\end{aligned}
$$

<u>Consider one case where explicitly solvable.</u> Let $f \equiv 0$, then (231) becomes

$$k_{xx} - k_{yy} = 0\,, \tag{234}$$

which has a general solution of the form

$$k(x,y) = \phi(x-y) + \psi(x+y). \tag{235}$$

From the boundary condition (233) we get

$$\phi(0) + \psi(2x) = 0\,, \tag{236}$$

which means that, without a loss of generality, we can set $\psi \equiv 0$ and $\phi(0) = 0$.

Therefore,

$$k(x,y) = \phi(x-y).$$

Substituting this expression into the boundary condition (232), we get

$$\phi'(x) = g(x) - \int_0^x \phi(x-y)g(y)\,dy\,. \tag{237}$$

Applying to this equation the Laplace transform with respect to x, we obtain

$$\begin{aligned} -s\phi(s) + \phi(0) &= g(s) - \phi(s)g(s) \\ \phi(s) &= \frac{g(s)}{g(s)-s}. \end{aligned} \tag{238}$$

Thus, for any function $g(x)$ one can obtain $k(x,y)$ in closed form.

Example 4 Let

$$g(x) = g.$$

Then

$$g(s) = \frac{g}{s}.$$

and from (238), $\phi(s)$ becomes

$$\phi(s) \quad = \quad \frac{g}{g - s^2} = -\sqrt{g}\frac{\sqrt{g}}{s^2 - g}.$$

This gives

$$\phi(z) = -\sqrt{g}\sinh(\sqrt{g}z)$$

and

$$k(x,y) = -\sqrt{g}\sinh(\sqrt{g}(x-y)).$$

Therefore, for the plant

$$
\begin{aligned}
u_t &= u_{xx} + g u(0) \\
u_x(0) &= 0
\end{aligned}
$$

the stabilizing controller is given by

$$u(1) = -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))u(y)\,dy\,.$$

Comparison with ODE Backstepping

Difference between ODE and PDE backstepping

A “finer” structural analogy where one might expect the z-system to be a spatial discretization of the w-system does not hold. If we discretize the PDE system $w_t = w_{xx}$, with boundary conditions $w(0,t) = w(1,t) = 0$, over a spatial grid with N points, we get the ODE system

$$\dot{w}_i = N^2(w_{i+1} - 2w_i + w_{i-1}),$$

which is different in structure from

$$\dot{z}_i = z_{i+1} - z_{i-1} - cz_i,$$

even after absorbing away the N^2 factor (into the time variable). This is where the subtle difference of the ODE backstepping and the PDE backstepping comes into play. The recursive procedure used for ODEs does not have a limit as the number of states goes to infinity. In contrast, the backstepping process for PDEs does have a limit.

Let us try to understand this difference by comparing the plant structure (81)–(83) with the plant structure $u_t = u_{xx} + \lambda u$. The former is dominated by a chain of integrators, while the latter is dominated by the diffusion operator. While the diffusion operator is a well-defined, meaningful object, an "infinite integrator chain" is not. It is for this reason that the infinite-dimensional backstepping design succeeds only if particular care is taken to convert the unstable parabolic PDE $u_t = u_{xx} + \lambda u$ into a stable target system $w_t = w_{xx}$ which is within the same PDE class, namely, parabolic.

To put it in simpler words, we make sure to retain the ∂_{xx} term in the target system, even though it may be tempting to go for some other target system, such as, for example, the first-order hyperbolic (transport equation-like) PDE $w_t = w_x - cw$, which is more reminiscent of the ODE target system (88)-(90). If such an attempt is made, the derivation of the PDE conditions for the kernel $k(x,y)$ would not be successful and the matching of terms between the plant $u_t = u_{xx} + \lambda u$ and the target system $w_t = w_{xx} - cw$ would result in terms that cannot be cancelled.

Meaning of the term *backstepping*

In the ODE setting this procedure is referred to as *integrator backstepping* because, as illustrated with the help of example (81)-(83), the design procedure propagates the feedback law synthesis "backwards" through a chain of integrators. Upon a careful inspection of the change of variables (84)–(86), the first "step" of the backstepping procedure is to treat the state y_2 as the control input in the subsystem $\dot{y}_1 = y_2 + y_1^3$, design the "control law" $y_2 = -y_1^3 - cy_1$, then "step back" through the integrator in the second subsystem $\dot{y}_2 = y_3 + y_2^3$ and design the "control" y_3 so that the error state $z_2 = y_2 - (-y_1^3 - cy_1)$ is forced to go zero, thus ensuring that the state y_2 acts (approximately) as the control $y_2 = -y_1^3 - cy_1$. This "backward stepping" through integrators continues until one encounters the actual control u in (87), which in the example (81)–(83) happens after two steps of backstepping.

Even though in our *continuum* version of backstepping for PDEs there are no simple integrators to step through, the analogy with the method for ODEs is in the triangularity of the change of variable and the pursuit of a stable target system. For this reason, we retain the term *backstepping* for PDEs.

Lower-triangular (strict-feedback) systems

Backstepping for ODEs is applicable to a fairly broad class of ODE systems which are referred to as *strict-feedback* systems. These systems are characterized by having a chain of integrators, the control appearing in the last equation, and additional terms (linear or nonlinear) having a "lower-triangular" structure. In this lower-triangular structure the first equation depends only on the first state, the term in the second equation depends on the first and the second states, and so on. In the example (81)–(83) the cubic terms had a "diagonal" dependence on the states y_i and thus, their structure was lower triangular and hence the plant (81)–(83) was of strict-feedback type. The change of variables (84)–(86) has a general lower triangular form.

The capability of backstepping to deal with lower-triangular ODE structures has motivated our extension of PDE backstepping from reaction-diffusion systems (which are of a "diagonal" kind) to the systems with lower-triangular strict-feedback terms $g(x)u(0,t)$ and $\int_0^x f(x,y)u(y,t)\,dy$. Such terms, besides being tractable by the backstepping method, happen to be essential in several applications, including flexible beams and Navier-Stokes equations.

Notes and References

The backstepping idea for PDEs appeared well before the development of finite-dimensional backstepping in the late 1980s.

Volterra operator transformations used for solving PDEs in

D. COLTON, *The solution of initial-boundary value problems for parabolic equations by the method of integral operators*, Journal of Differential Equations, 26 (**1977**), pp. 181–190.

and for developing controllability results in

T. I. SEIDMAN, *Two results on exact boundary control of parabolic equations*, Applied Mathematics and Optimization, 11 (**1984**), pp. 145–152.

Homework

1. For the plant

$$
\begin{aligned}
u_t &= u_{xx} + \lambda u \\
u_x(0) &= 0
\end{aligned}
$$

design the Neumann stabilizing controller ($u_x(1)$ actuated).

Hint: use the target system

$$
\begin{aligned}
w_t &= w_{xx} \\
w_x(0) &= 0 \\
w_x(1) &= -\frac{1}{2} w(1)\,.
\end{aligned} \tag{239}
$$

This system is asymptotically stable. Note also that you do not need to find $k(x,y)$, it has already been found in Example 3. You only need to use the condition (239) to derive the controller.

2. Find the PDE for the kernel $l(x,y)$ of the inverse transformation

$$
u(x) = w(x) + \int_0^x l(x,y) w(y)\, dy\,,
$$

which relates the systems u and w from Exercise 1. By comparison with the PDE for $k(x,y)$, show that

$$l(x,y) = -\lambda x \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}.$$

3. Design the Dirichlet boundary controller for the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u_x(0) &= -qu(0) \end{aligned}$$

Follow these steps:

1) Use the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy \tag{240}$$

to map the plant into the target system

$$\begin{aligned} w_t &= w_{xx} && (241)\\ w_x(0) &= 0 && (242)\\ w(1) &= 0\,. && (243) \end{aligned}$$

Show that $k(x,y)$ satisfies the following PDE:

$$k_{xx}(x,y) = k_{yy}(x,y) \tag{244}$$
$$k_y(x,0) = -qk(x,0) \tag{245}$$
$$k(x,x) = -q\,. \tag{246}$$

2) The general solution of the PDE (244) has the form $k(x,y) = \phi(x-y) + \psi(x+y)$, where ϕ and ψ are arbitrary functions. Using (246) it can be shown that $\psi \equiv 0$. Find ϕ from the conditions (245) and (246). Write the solution for $k(x,y)$.

3) Write down the controller.

4. Show that the solution of the closed-loop system from Exercise 3 is ($\sigma_n = \pi(2n+1)/2$)

$$u(x,t) = 2\sum_{n=0}^{\infty} e^{-\sigma_n^2 t}\left(\sigma_n\cos(\sigma_n x) - q\sin(\sigma_n x)\right) \times \int_0^1 \frac{\sigma_n\cos(\sigma_n\xi) - q\sin(\sigma_n\xi) + (-1)^n q e^{q(1-\xi)}}{\sigma_n^2 + q^2} u_0(\xi)\,d\xi\,.$$

To do this, first write the solution of the system (241)–(243). Then use the transformation (240) with the $k(x,y)$ that you found in Exercise 3 to express the initial condition $w_0(x)$ in terms of $u_0(x)$ (you will need to change the order of integration in one of the terms to do this). Finally, write the solution for $u(x,t)$ using the inverse transformation

$$u(x) = w(x) - q\int_0^x w(y)\,dy$$

(i.e., $l(x,y) = -q$ in this problem; feel free to prove it).

Note that it is not possible to write a closed form solution for the open loop plant, but it is possible to do so for the closed loop system!

5. For the plant

$$u_t = u_{xx} + bu_x + \lambda u$$
$$u_x(0) = -\frac{b}{2}u(0)$$

design the Neumann stabilizing controller ($u_x(1)$ actuated).

Hint: by transforming the plant to a system without b-term, reduce the problem to Exercise 1.

6. For the plant

$$u_t = u_{xx} + 3e^{2x}u(0) \tag{247}$$

$$u_x(0) = 0 \tag{248}$$

design the Dirichlet stabilizing controller.

Observer Design

Sensors placed at the boundaries.

Motivation: fluid flows (aerodynamics, acoustics, chemical process control, etc.).

Observer Design for PDEs with Boundary Sensing

$$\begin{aligned} u_t &= u_{xx} + \lambda u && (249)\\ u_x(0) &= 0 && (250)\\ u(1) &= U(t) \qquad \text{(open-loop or feedback signal)} && (251)\end{aligned}$$

$$\text{meas. output} = u(0) \qquad \text{(at the boundary w/ Neumann b.c.)} \tag{252}$$

Observer:

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda \hat{u} + p_1(x)[u(0) - \hat{u}(0))] && (253)\\ \hat{u}_x(0) &= p_{10}[u(0) - \hat{u}(0)] && (254)\\ \hat{u}(1) &= U(t) && (255)\end{aligned}$$

The function $p_1(x)$ and the constant p_{10} are observer gains to be determined.

Mimics the finite-dimensional observer format of "copy of the plant plus output injection."

Finite-dim plant

$$\begin{aligned} \dot{x} &= Ax + Bu && (256) \\ y &= Cx && (257) \end{aligned}$$

Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad (258)$$

L = observer gain
$L(y - C\hat{x})$ = "output error injection"

In (253), (254) the obs. gains $p_1(x)$ and p_{10} form an inf-dim "vector" like L.

Objective: find $p_1(x)$ and p_{10} such that $\hat{u}$ converges to u.

Error variable

$$\tilde{u} = u - \hat{u} \tag{259}$$

Error system

$$\begin{aligned}
\tilde{u}_t &= \tilde{u}_{xx} + \lambda\tilde{u} - p_1(x)\tilde{u}(0) && (260)\\
\tilde{u}_x(0) &= -p_{10}\tilde{u}(0) && (261)\\
\tilde{u}(1) &= 0 && (262)
\end{aligned}$$

Magic needed: remove the destabilizing term $\lambda\tilde{u}(x)$ using feedback of boundary term $\tilde{u}(0)$

Backstepping transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)dy \tag{263}$$

Target system

$$\tilde{w}_t = \tilde{w}_{xx} \tag{264}$$

$$\tilde{w}_x(0) = 0 \tag{265}$$

$$\tilde{w}(1) = 0 \tag{266}$$

Differentiating the transformation (303), we get

$$
\begin{aligned}
\tilde{u}_t(x) &= \tilde{w}_t(x) - \int_0^x p(x,y)\tilde{w}_{yy}(y)\,dy \\
&= \tilde{w}_t(x) - p(x,x)\tilde{w}_x(x) + p(x,0)\tilde{w}_x(0) + p_y(x,x)\tilde{w}(x) \\
&\quad - p_y(x,0)\tilde{w}(0) - \int_0^x p_{yy}(x,y)\tilde{w}(y)\,dy\,, \qquad (267)\\
\tilde{u}_{xx}(x) &= \tilde{w}_{xx}(x) - \tilde{w}(x)\frac{d}{dx}p(x,x) - p(x,x)\tilde{w}_x(x) \\
&\quad - p_x(x,x)\tilde{w}(x) - \int_0^x p_{xx}(x,y)\tilde{w}(y)\,dy\,. \qquad (268)
\end{aligned}
$$

Subtracting (268) from (267), we obtain:

$$\begin{aligned}
\tilde{u}_t - \tilde{u}_{xx} &= 2\tilde{w}(x)\frac{d}{dx}p(x,x) - p_y(x,0)\tilde{w}(0) + \int_0^x (p_{xx}(x,y) - p_{yy}(x,y))\tilde{w}(y)\,dy \\
&= \lambda \underbrace{\left(\tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)\,dy\right)}_{\tilde{u}} - p_1(x)\underbrace{\tilde{w}(0)}_{\tilde{u}(0)} \qquad (269)
\end{aligned}$$

For the last equality to hold, three conditions must be satisfied:

$$p_{xx}(x,y) - p_{yy}(x,y) = -\lambda p(x,y) \qquad (270)$$

$$\frac{d}{dx}p(x,x) = \frac{\lambda}{2} \qquad (271)$$

$$p_1(x) = p_y(x,0) \qquad (272)$$

Recall the backstepping transform

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)dy \tag{273}$$

$$\tilde{u}_x(x) = \tilde{w}_x(x) - p(x,x)\tilde{w}(x) - \int_0^x p_x(x,y)\tilde{w}(y)dy \tag{274}$$

and set $x = 1$ and $x = 0$:

$$\tilde{u}(0) = \tilde{w}(0) \tag{275}$$

$$\tilde{u}(1) = \tilde{w}(1) - \int_0^1 p(1,y)\tilde{w}(y)dy \tag{276}$$

$$\tilde{u}_x(0) = \tilde{w}_x(0) - p(0,0)\tilde{w}(0) \tag{277}$$

Recall that the target system requires that

$$\tilde{w}_x(0) = 0 \tag{278}$$

$$\tilde{w}(1) = 0 \tag{279}$$

It follows that

$$
\begin{aligned}
\tilde{u}(1) &= -\int_0^1 p(1,y)\tilde{w}(y)dy && (280) \\
\tilde{u}_x(0) &= -p(0,0)\tilde{u}(0) && (281)
\end{aligned}
$$

Recall now the boundary conditions (261), (262)

$$
\begin{aligned}
\tilde{u}_x(0) &= -p_{10}\tilde{u}(0) && (282) \\
\tilde{u}(1) &= 0 && (283)
\end{aligned}
$$

This provides the conditions:

$$
\begin{aligned}
p_{10} &= p(0,0) && (284) \\
p(1,y) &= 0 && (285)
\end{aligned}
$$

Let us solve (271) and (285) for $p(x,x)$ and combine the result with the equations (270) and (285):

$$\boxed{\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) = & \; -\lambda p(x,y) \\ p(1,y) = & \; 0 \\ p(x,x) = & \; \frac{\lambda}{2}(x-1) \end{aligned}} \tag{286}$$

To solve, make a change of variables

$$x = 1 - y, \quad y = 1 - x, \quad p(x,y) = p(x,y) \tag{287}$$

which gives the following PDE:

$$p_{xx}(x,y) - p_{yy}(x,y) = \lambda p(x,y) \tag{288}$$

$$p(x,0) = 0, \tag{289}$$

$$p(x,x) = -\frac{\lambda}{2}x. \tag{290}$$

The solution is

$$p(x,y) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}}. \tag{291}$$

or, in the original variables,

$$p(x,y) = -\lambda(1-x)\frac{I_1(\sqrt{\lambda(2-x-y)(x-y)})}{\sqrt{\lambda(2-x-y)(x-y)}}. \tag{292}$$

The observer gains,obtained using (272) and (284) are

$$p_1(x) = p_y(x,0) = \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) \tag{293}$$

$$p_{10} = p(0,0) = -\frac{\lambda}{2}. \tag{294}$$

Summary of the plant and observer

Plant
$$u_t = u_{xx} + \lambda u \tag{295}$$
$$u_x(0) = 0 \tag{296}$$
$$u(1) = U \tag{297}$$

Observer
$$\hat{u}_t = \hat{u}_{xx} + \lambda\hat{u} \boxed{+\frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right) \boxed{[u(0) - \hat{u}(0)]}} \tag{298}$$
$$\hat{u}_x(0) = \boxed{-\frac{\lambda}{2}\boxed{[u(0) - \hat{u}(0)]}} \tag{299}$$
$$\hat{u}(1) = U \tag{300}$$

Output Feedback

The observer can be used with any controller.

For linear systems, the separation principle (or "certainty equivalence") holds, i.e. the combination of a separately designed state feedback controller and observer results in a stabilizing output-feedback controller.

Next, we establish the separation principle for our observer-based output feedback design.

The control backstepping transformation $\hat{u} \mapsto \hat{w}$ (on the state estimate)

$$\hat{w}(x) = \hat{u}(x) - \int_0^x k(x,y)\hat{u}(y)\,dy \qquad \text{(direct)} \tag{301}$$

$$\hat{u}(x) = \hat{w}(x) + \int_0^x l(x,y)\hat{w}(y)\,dy \qquad \text{(inverse)} \tag{302}$$

and the observer backstepping transformation $\tilde{u} \mapsto \tilde{w}$

$$\tilde{u}(x) = \tilde{w}(x) - \int_0^x p(x,y)\tilde{w}(y)dy \qquad \text{(inverse)} \tag{303}$$

map the closed-loop sys into a target system **of cascade form** $\boxed{\tilde{w} \longrightarrow \hat{w}}$

$$\hat{w}_t = \hat{w}_{xx} + \left\{ p_1(x) - \int_0^x k(x,y)p_1(y)\,dy + p_{10}k(x,0) \right\} \tilde{w}(0) \tag{304}$$

$$\hat{w}_x(0) = p_{10}\tilde{w}(0) \tag{305}$$

$$\hat{w}(1) = 0 \tag{306}$$

$$\tilde{w}_t = \tilde{w}_{xx} \tag{307}$$

$$\tilde{w}_x(0) = 0 \qquad \textbf{AUTONOMOUS SYST.} \tag{308}$$

$$\tilde{w}(1) = 0, \tag{309}$$

where $k(x,y)$ is the kernel of the control transformation and $p_1(x)$, p_{10} are observer gains.

The $\tilde{w}$-system and the homogeneous part of the $\hat{w}$-system (without $\tilde{w}(0,t)$) are exponentially stable heat equations.

To show that the system ($\hat{w}$, $\tilde{w}$) is exponentially stable, we use the weighted Lyapunov function

$$V = \frac{A}{2}\int_0^1 \tilde{w}(x)^2\,dx + \frac{1}{2}\int_0^1 \hat{w}(x)^2\,dx\,, \tag{310}$$

where A is the weighting constant to be chosen later.

Taking the time derivative of (310), we get

$$\begin{aligned}
\dot{V} \;&=\; -A\int_0^1 \tilde{w}_x(x)^2\,dx - \int_0^1 \hat{w}_x(x)^2\,dx \qquad\qquad \textbf{GOOD}\\
&\quad -p_{10}\hat{w}(0)\tilde{w}(0) \qquad\qquad \textbf{BAD}\\
&\quad +\tilde{w}(0)\int_0^1 \hat{w}(x)\left\{p_1(x) - \int_0^x k(x,y)p_1(y)\,dy + p_{10}k(x,0)\right\}dx \qquad\qquad \textbf{BAD}
\end{aligned}$$

Using the Young and Agmon inequalities, we estimate

$$-p_{10}\hat{w}(0)\tilde{w}(0) \;\; \leq \;\; \frac{1}{4}\hat{w}(0)^2 + p_{10}^2\tilde{w}(0)^2 \leq \frac{1}{4}\int_0^1 \hat{w}_x(x)^2\,dx + p_{10}^2\int_0^1 \tilde{w}_x(x)^2\,dx$$

and

$$\begin{aligned} &\tilde{w}(0)\int_0^1 \hat{w}(x)\left\{p_1(x) - \int_0^x k(x,y)p_1(y)\,dy + p_{10}k(x,0)\right\}dx \\ &\leq \frac{1}{4}\int_0^1 \hat{w}_x(x)^2\,dx + B^2\int_0^1 \tilde{w}_x(x)^2\,dx \end{aligned}$$

where $B = \max_{x\in[0,1]}\{p_1(x) - \int_0^x k(x,y)p_1(y)\,dy + p_{10}k(x,0)\}$.

With these estimates, we obtain

$$\begin{aligned} \dot{V} &\leq -(A - B^2 - p_{10}^2)\int_0^1 \tilde{w}_x(x)^2\,dx - \frac{1}{2}\int_0^1 \hat{w}_x(x)^2\,dx \\ &\leq -\frac{1}{4}(A - B^2 - p_{10}^2)\int_0^1 \tilde{w}(x)^2\,dx - \frac{1}{8}\int_0^1 \hat{w}(x)^2\,dx \qquad \text{(Poincare)} \end{aligned}$$

Taking $A = 2(B^2 + p_{10}^2)$, we get

$$\dot{V} \;\; \leq \;\; -\frac{1}{4}V$$

Hence, the system $(\hat{w}, \tilde{w})$ is exponentially stable.

The system $(\hat{u}, \tilde{u})$ is also exponentially stable since it is related to $(\hat{w}, \tilde{w})$ by the invertible coordinate transformations (303) and (302).

We have proved the **separation principle**.

Output feedback design for **anti-collocated** setup

Plant
$$u_t = u_{xx} + \lambda u \tag{311}$$
$$u_x(0) = 0 \tag{312}$$

Observer
$$\hat{u}_t = \hat{u}_{xx} + \lambda \hat{u} + \frac{\lambda(1-x)}{x(2-x)} I_2\left(\sqrt{\lambda x(2-x)}\right)[u(0) - \hat{u}(0)] \tag{313}$$
$$\hat{u}_x(0) = -\frac{\lambda}{2}[u(0) - \hat{u}(0)] \tag{314}$$
$$\hat{u}(1) = -\int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \hat{u}(y)\, dy \tag{315}$$

Controller
$$u(1) = -\int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \hat{u}(y)\, dy \tag{316}$$

Observer Design for Collocated Sensor and Actuator

$$\begin{aligned} u_t &= u_{xx} + \lambda u && (317) \\ u_x(0) &= 0 && (318) \\ u(\mathbf{1}) &= U(t) && (319) \\ u_x(\mathbf{1}) &- \text{measurement} \end{aligned}$$

Observer

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + \lambda\hat{u} + p_1(x)[u_x(1) - \hat{u}_x(1)] && (320) \\ \hat{u}_x(0) &= 0 && (321) \\ \hat{u}(1) &= U(t) + p_{10}[u_x(1) - \hat{u}_x(1)] && (322) \end{aligned}$$

Error $\tilde{u} = u - \hat{u}$, error system

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx} + \lambda\tilde{u} - p_1(x)\tilde{u}_x(1) && (323) \\ \tilde{u}_x(0) &= 0 && (324) \\ \tilde{u}(1) &= -p_{10}\tilde{u}_x(1) && (325) \end{aligned}$$

Backstepping transformation

$$\boxed{\tilde{u}(x) = \tilde{w}(x) - \int_x^1 p(x,y)\tilde{w}(y)dy} \tag{326}$$

to convert the error system into target system:

$$\tilde{w}_t = \tilde{w}_{xx} \tag{327}$$

$$\tilde{w}_x(0) = 0 \tag{328}$$

$$\tilde{w}(1) = 0. \tag{329}$$

Note that the integral in the transformation runs <u>from x to 1</u> instead of the usual 0 to x!

We get the kernel PDE

$$
\begin{aligned}
p_{xx}(x,y) - p_{yy}(x,y) &= -\lambda p(x,y) && (330)\\
p_x(0,y) &= 0, && (331)\\
p(x,x) &= -\frac{\lambda}{2}x && (332)
\end{aligned}
$$

From the resulting target system

$$
\begin{aligned}
\tilde{w}_t &= \tilde{w}_{xx} + [p(x,1) - p_1(x)]\tilde{w}_x(1) && (333)\\
\tilde{w}_x(0) &= 0 && (334)\\
\tilde{w}(1) &= -p_{10}w_x(1) && (335)
\end{aligned}
$$

the observer gains should be chosen as

$$
p_1(x) = p(x,1), \qquad p_{10} = 0. \tag{336}
$$

To solve the kernel PDE (330)–(332) we introduce the change of variables

$$x = y, \quad y = x, \quad p(x,y) = p(x,y)$$

to get

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= \lambda p(x,y) && (337) \\ p_y(x,0) &= 0, && (338) \\ p(x,x) &= -\frac{\lambda}{2}x\,. && (339) \end{aligned}$$

This PDE's solution is

$$\begin{aligned} p(x,y) &= -\lambda x \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)}} \\ &= -\lambda y \frac{I_1(\sqrt{\lambda(y^2-x^2)})}{\sqrt{\lambda(y^2-x^2)}} \end{aligned}$$

Therefore, the observer gains are

$$p_1(x) = -\lambda \frac{I_1(\sqrt{\lambda(1-x^2)})}{\sqrt{\lambda(1-x^2)}} \tag{340}$$

and $p_{10} = 0$.

Output feedback design for collocated setup

Plant $$u_t = u_{xx} + \lambda u \tag{341}$$

$$u_x(0) = 0 \tag{342}$$

Observer $$\hat{u}_t = \hat{u}_{xx} + \lambda \hat{u} - \lambda \frac{I_1(\sqrt{\lambda(1-x^2)})}{\sqrt{\lambda(1-x^2)}}[u_x(1) - \hat{u}_x(1)] \tag{343}$$

$$\hat{u}_x(0) = 0 \tag{344}$$

$$\hat{u}(1) = -\int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \hat{u}(y)\,dy \tag{345}$$

Controller $$u(1) = -\int_0^1 \lambda \frac{I_1(\sqrt{\lambda(1-y^2)})}{\sqrt{\lambda(1-y^2)}} \hat{u}(y)\,dy \tag{346}$$

The fact that $p_1(x) = k(1,x)$ demonstrates the duality between observer and control designs, the property known from the finite-dimensional designs for linear systems.

We used the same decay rates for the observer and controller. One can easily modify the designs to make the observer faster that the controller.

Compensator Transfer Function

When both the controller and the observer are given explicitly, one can derive a frequency domain representation of the compensator.

To illustrate this, consider the following PDE:

$$
\begin{aligned}
u_t &= u_{xx} + g u(0) &\quad (347)\\
u_x(0) &= 0 &\quad (348)
\end{aligned}
$$

with

$$
\begin{aligned}
u(1) &= \text{input}\\
u(0) &= \text{output}
\end{aligned}
$$

We first derive the transfer function of the open-loop plant.

Taking the Laplace transform of (347), (348) we get

$$
\begin{aligned}
su(x,s) &= u''(x,s) + gu(0,s) && (349)\\
u'(0,s) &= 0 && (350)
\end{aligned}
$$

The general solution for this second order ODE in x is given by

$$u(x,s) = A\sinh(\sqrt{s}x) + B\cosh(\sqrt{s}x) + \frac{g}{s}u(0,s)\,, \tag{351}$$

where A and B are to be determined. From the boundary condition (350) we have

$$u'(0,s) = A\sqrt{s} = 0 \Rightarrow A = 0\,. \tag{352}$$

By setting $x = 0$ in (351), we find B:

$$B = u(0,s)\left(1 - \frac{g}{s}\right)\,. \tag{353}$$

Hence, we get

$$u(x,s) = u(0,s)\left[\frac{g}{s} + \left(1 - \frac{g}{s}\right)\cosh(\sqrt{s}x)\right]\,.$$

Setting $x = 1$ we obtain the **plant transfer function**

$$\boxed{u(0,s) = \frac{s}{g + (s-g)\cosh(\sqrt{s})}u(1,s)} \tag{354}$$

This plant has no zeros and infinite relative degree.

Using a Taylor expansion of the $\cosh$ term, we get an approximate expression for the plant transfer function,

$$\frac{u(0,s)}{u(1,s)} \approx \frac{1}{1 - \frac{g}{2} + \left(\frac{1}{2} - \frac{g}{4!}\right) s + \left(\frac{1}{4!} - \frac{g}{6!}\right) s^2 + \dots}. \tag{355}$$

Let us now derive the frequency domain representation of the compensator.

The observer PDE is given by

$$\begin{aligned} \hat{u}_t &= \hat{u}_{xx} + g u(0) && (356) \\ \hat{u}_x(0) &= 0 && (357) \\ \hat{u}(1) &= -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))\hat{u}(y)dy\,. && (358) \end{aligned}$$

Applying the Laplace transform, we get

$$\begin{aligned} s\hat{u}(x,s) &= \hat{u}''(x,s) + g u(0,s) && (359) \\ \hat{u}'(0,s) &= 0 && (360) \\ \hat{u}(1,s) &= -\int_0^1 \sqrt{g}\sinh(\sqrt{g}(1-y))\hat{u}(y,s)dy && (361) \end{aligned}$$

The general solution of PDE (359) with boundary condition (360) is given by

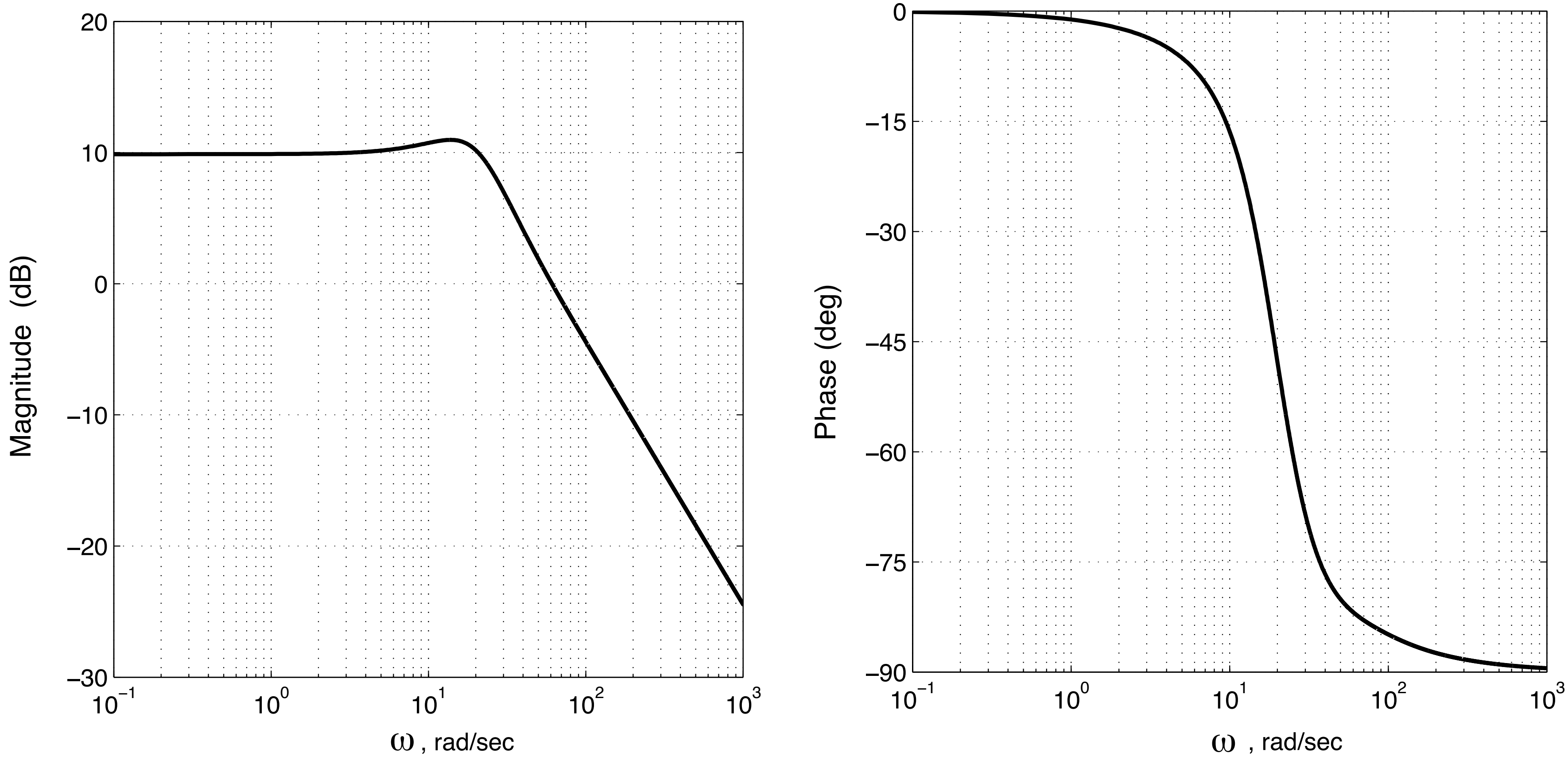
$$\hat{u}(x,s) = \hat{u}(0,s)\cosh(\sqrt{s}x) + \frac{g}{s}\left(1 - \cosh(\sqrt{s}x)\right)u(0,s). \tag{362}$$

Substituting (362) into (361), and evaluating the integral, we express $\hat{u}(0,s)$ as a function of $u(0,s)$:

$$\hat{u}(0,s) = \frac{\cosh(\sqrt{s}) - \cosh(\sqrt{g})}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})}gu(0,s)\,. \tag{363}$$

Setting $x = 1$ in (362) and using (363) we get the **transfer function of the compensator**

$$\boxed{u(1,s) = \frac{g}{s}\left(-1 + \frac{(s-g)\cosh(\sqrt{s})\cosh(\sqrt{g})}{s\cosh(\sqrt{s}) - g\cosh(\sqrt{g})}\right)u(0,s)} \tag{364}$$



Bode plots of the compensator (364)

Compensator can be approximated by a second order, relative degree one transfer function, for example,

$$C(s) \approx 60 \frac{s+17}{s^2+25s+320}. \tag{365}$$

Homework

1. Design an observer for the following system:

$$\begin{aligned} u_t &= u_{xx} \\ u_x(0) &= -qu(0) \\ u(1) &= U(t) \end{aligned}$$

with only $u_x(1)$ available for measurement.

Follow these steps:

1) Write down the observer for this system, with output injection entering the PDE and the boundary condition at $x = 1$.

2) Use the transformation

$$\tilde{u}(x) = \tilde{w}(x) - \int_x^1 p(x,y)\tilde{w}(y)\,dy \tag{366}$$

to map the error system into the target system

$$\begin{aligned} \tilde{w}_t &= \tilde{w}_{xx} && (367) \\ \tilde{w}_x(0) &= 0 && (368) \\ \tilde{w}(1) &= 0\,. && (369) \end{aligned}$$

Show that $p(x,y)$ satisfies the PDE

$$p_{xx}(x,y) = p_{yy}(x,y) \tag{370}$$
$$p_x(0,y) = -qp(0,y) \tag{371}$$
$$p(x,x) = -q\,. \tag{372}$$

and that the observer gains are given by $p_{10}=0$ and $p_1(x) = p(x,1)$.

3) Solve the PDE for $p(x,y)$ (look for the solution in the form $p(x,y) = \phi(y-x)$). Find $p_1(x)$.

2. Find the frequency domain representation of the plant

$$\begin{aligned} u_t &= u_{xx} \\ u_x(0) &= -qu(0) \\ u(1) &= U(t) \end{aligned}$$

with $u(0)$ measured and $u(1)$ actuated, i.e., find $G(s)$ such that $u(0,s) = G(s)U(s)$.

First Order Hyperbolic PDEs and Delay Equations

First Order Hyperbolic PDEs

$$u_t = u_x + \boxed{g(x)u(0) + \int_0^x f(x,y)u(y)dy} \tag{373}$$

$$u(1) = \text{control} \tag{374}$$

Unlike in second order PDEs, here we specify only one boundary condition.

Backstepping transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)dy \tag{375}$$

Target system

$$w_t = w_x \tag{376}$$

$$w(1) = 0 \tag{377}$$

Ddelay line with delay $= 1$, output $w(0,t) = w(1,t-1)$, and zero input at $w(1,t)$.

Akin to traffic flow over a stretch of a road with no new cars permitted to enter after $t = 0$.

Explicit solution

$$w(x,t) = \begin{cases} w_0(t+x) & 0 \le t+x < 1 \\ 0 & t+x \ge 1, \end{cases} \tag{378}$$

where $w_0(x)$ is the initial condition.

Solution becomes zero in time $= 1$.

Kernel PDE from (373)–(377):

$$k_x(x,y)+k_y(x,y) = \int_y^x k(x,\xi)f(\xi,y)d\xi - f(x,y) \tag{379}$$

$$k(x,0) = \int_0^x k(x,y)g(y)dy - g(x) \tag{380}$$

First-order hyperbolic. Only one boundary condition.

Controller (as before)

$$u(1) = \int_0^1 k(1,y)u(y)\,dy \tag{381}$$

Example 5

$$u_t = u_x + ge^{bx}u(0) \tag{382}$$

Kernel PDE (379) becomes

$$k_x + k_y = 0\,, \tag{383}$$

which has a general solution

$$k(x,y) = \phi(x-y)$$

If we plug this solution into (380), we get the integral equation

$$\phi(x) = \int_0^x ge^{by}\phi(x-y)dy - ge^{bx}\,. \tag{384}$$

Solution obtained by Laplace transform in x:

$$\phi(s) = -\frac{g}{s-(b+g)}\,. \tag{385}$$

After taking the inverse Laplace transform, $\phi(x) = -ge^{(b+g)x}$, and hence

$$\boxed{k(x,y) = -ge^{(b+g)(x-y)}}$$

Example 6 [TRICKIER–can skip]

$$u_t = u_x + \int_0^x f e^{b(x-y)} u(y)\, dy \tag{386}$$

Kernel PDE

$$k_x + k_y = \int_y^x k(x,\xi) f e^{b(\xi-y)} d\xi - f e^{b(x-y)} \tag{387}$$

$$k(x,0) = 0\,. \tag{388}$$

After we differentiate (387) with respect to y, the integral term is eliminated:

$$k_{xy} + k_{yy} = -fk - bk_x - bk_y\,. \tag{389}$$

Since we now increased the order of the equation, we need an extra boundary condition, which we get by setting $y = x$ in (387):

$$\frac{d}{dx}k(x,x) = k_x(x,x) + k_y(x,x) = -f \qquad \Longrightarrow \qquad k(x,x) = -fx\,. \tag{390}$$

Introducing the change of variables

$$\boxed{k(x,y) = p(z,y)e^{b(z-y)/2}, \qquad z = 2x - y\,,} \tag{391}$$

we get the following PDE for $p(z,y)$:

$$p_{zz}(z,y) - p_{yy}(z,y) = f\,p(z,y) \tag{392}$$

$$p(z,0) = 0 \tag{393}$$

$$p(z,z) = -fz\,. \tag{394}$$

Solution:

$$p(z,y) = -2fy\frac{I_1\left(\sqrt{f(z^2-y^2)}\right)}{\sqrt{f(z^2-y^2)}} \tag{395}$$

or, in the original variables,

$$\boxed{\boxed{k(x,y) = -fe^{b(x-y)}y\frac{I_1\left(2\sqrt{fx(x-y)}\right)}{\sqrt{fx(x-y)}}\,.}} \tag{396}$$

ODE Systems with Actuator Delay

$$\dot{X} = AX + BU(t-D)\,, \tag{397}$$

$(A,B) =$ stabilizable pair
$D =$ delay on input signal $U(t)$

First-order hyperbolic PDE as a model for the delay

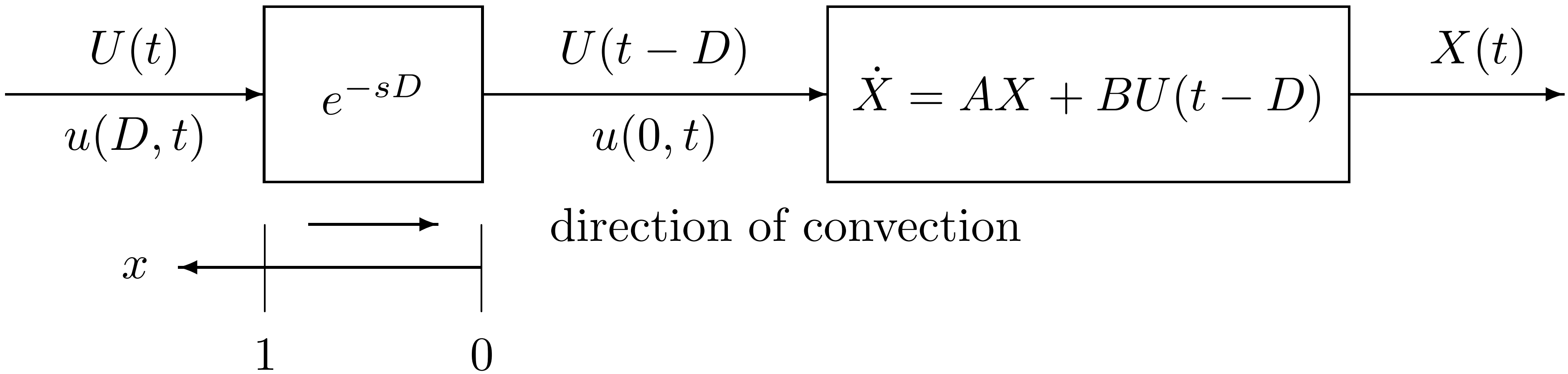
$$\begin{aligned} u_t &= u_x & (398) \\ u(D,t) &= U(t) & (399) \end{aligned}$$

The solution is $\boxed{u(x,t) = U(t+x-D)}$ and therefore $u(0,t) = U(t-D)$.

The system (397) can be now written as the ODE

$$\dot{X} = AX + Bu(0,t) \tag{400}$$

along with the PDE (398), (399).



$$\begin{aligned} \dot{X} &= AX + Bu(0,t) \\ u_t &= u_x \\ u(D,t) &= U(t) \end{aligned}$$

Suppose a stabilizing static state-feedback controller

$$U = KX$$

has been designed for the system without delay ($D = 0$), i.e.,

$$(A + BK) = \text{Hurwitz matrix}$$

When $D \neq 0$, we choose the target system as

$$\begin{aligned} \dot{X} &= (A + BK)X + Bw(0) && (401) \\ w_t &= w_x && (402) \\ w(D) &= 0\,. && (403) \end{aligned}$$

w becomes zero D time units, and after that the ODE (401) decays exponentially by the nominal design.

To map (398)-(400) into (401)–(403), consider the backstepping transformation

$$w(x) = u(x) - \int_0^x q(x,y)u(y)dy - \gamma(x)^T X\,, \tag{404}$$

where $q(x,y)$ and $\gamma(x)$ are to be designed.

$$\begin{aligned} w_x &= u_x - q(x,x)u(x) - \int_0^x q_x(x,y)u(y)dy - \gamma'(x)^T X \qquad (405)\\ w_t &= u_t - \int_0^x q(x,y)u_t(y)dy - \gamma(x)^T\left[AX + Bu(0)\right] \\ &= u_x - q(x,x)u(x) + q(x,0)u(0) + \int_0^x q_y(x,y)u(y)dy \\ &\quad -\gamma(x)^T\left[AX + Bu(0)\right]. \qquad (406) \end{aligned}$$

Subtracting (405) from (406) we get

$$\begin{aligned} \int_0^x (q_x(x,y) + q_y(x,y))u(y)dy + \left[q(x,0) - \gamma(x)^T B\right]u(0) \\ + \left[\gamma'(x)^T - \gamma(x)^T A\right] X \;=\; 0\,. \end{aligned} \tag{407}$$

We get three conditions

$$
\begin{aligned}
q_x(x,y)+q_y(x,y) &= 0 && \text{[PDE]} && (408)\\
q(x,0) &= \gamma(x)^T B && \text{[boundary condition]} && (409)\\
\gamma'(x) &= A^T\gamma(x) && \text{[ODE]} && (410)
\end{aligned}
$$

To find the initial condition for the ODE, let us set $x=0$ in (404), which gives

$$w(0)=u(0)-\gamma(0)^T X$$

Substituting this expression into (401), we get

$$\dot{X} \;=\; AX+Bu(0)+B\left(K-\gamma(0)^T\right)X\,. \tag{411}$$

Comparing this equation with (400), we have

$$\gamma(0)=K^T$$

Therefore the solution to the ODE (410) is $\gamma(x)=e^{A^T x}K^T$ which gives

$$\boxed{\gamma(x)^T = Ke^{Ax}} \tag{412}$$

A general solution to (408) is $q(x,y) = \phi(x-y)$, where ϕ is determined from (409).

We get the PDE solution as

$$\boxed{q(x,y) = Ke^{A(x-y)}B} \tag{413}$$

Substitute the gains $\gamma(x)$ and $q(x,y)$ into the transformation (404) and set $x = D$ to get the control law:

$$\boxed{\boxed{u(D) = K\int_0^D e^{A(D-y)}Bu(y)dy + Ke^{AD}X}} \tag{414}$$

The above controller is given in terms of the transport delay state $u(y)$.

Using (398)–(399), derive the representation in terms of the input signal $U(t)$:

$$\boxed{U(t) = K\left[e^{AD}X + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)d\theta\right]} \tag{415}$$

Notice that

- This is an infinite dimensional controller (input depends on its history over $[t-D,t]$).
- $U(t) = KX(t+D)$ [predictor-based feedback]
- Controller works harder than $U(t) = KX(t)$ when A is unstable.

Notes and References

Controller (415) draws inspiration from the Smith Predictor (1957). However, SP is not applicable to unstable plants

Controller (415) first proposed by Mayne (1968).

Most people think that it was introduced by Manitius and Olbrot (1978) as Finite Spectrum Assignment.

Generalized by Artstein (1982) to time-varying plants and distributed delays.

Lyapunov stability first proved by Krstic and Smyshlyaev (2008).

Homework

1. Derive the kernel PDE (379)–(380).

   Hint: use the formula

$$\int_0^x \int_0^\xi k(x,\xi) f(\xi,y) u(y)\, dy\, d\xi = \int_0^x \int_y^x k(x,\xi) f(\xi,y) u(y)\, d\xi\, dy$$

2. Consider the system

$$\begin{aligned} \dot{X} &= AX \\ Y(t) &= CX(t-D)\,, \end{aligned}$$

   where the output equation can be also represented as

$$\begin{aligned} u_t &= u_x \\ u(D,t) &= CX(t) \\ Y(t) &= u(0,t)\,. \end{aligned}$$

Introduce the observer

$$
\begin{aligned}
\dot{\hat{X}} &= A\hat{X} + e^{AD}L\left(Y(t) - \hat{u}(0,t)\right) \\
\hat{u}_t &= \hat{u}_x + Ce^{Ax}L\left(Y(t) - \hat{u}(0,t)\right) \\
\hat{u}(D,t) &= C\hat{X}(t)
\end{aligned}
$$

where L is chosen such that $A - LC$ is Hurwitz. Show that the transformation

$$
\tilde{w}(x) = \tilde{u}(x) - Ce^{A(x-D)}\tilde{X}\,,
$$

where $\tilde{X} = X - \hat{X}$, $\tilde{u} = u - \hat{u}$, converts the $(\tilde{X}, \tilde{u})$ system into

$$
\begin{aligned}
\dot{\tilde{X}} &= \left(A - e^{AD}LCe^{-AD}\right)\tilde{X} - e^{AD}L\tilde{w}(0) \\
\tilde{w}_t &= \tilde{w}_x \\
\tilde{w}(D) &= 0
\end{aligned}
$$

Note that the $\tilde{w}$ system is exponentially stable and that the matrix $A - e^{AD}LCe^{-AD}$ is Hurwitz (you can see this by using a similarity transformation e^{AD}, and using the fact that it commutes with A).

3. Show that the observer in Exercise 2 can be represented as

$$\begin{aligned}\dot{\hat{X}} &= A\hat{X} + e^{AD}L(Y-\hat{Y}) \\ \hat{Y}(t) &= C\hat{X}(t-D) + C\int_{t-D}^{t} e^{A(t-\theta)}L(Y(\theta)-\hat{Y}(\theta))\,d\theta\end{aligned}$$

Hint: take a Laplace transform of the $\hat{u}(x,t)$ system with respect to t; solve the resulting first order ODE w.r.t x with $\hat{u}(0,s) = \hat{Y}(s)$ as initial condition and $Y(s)-\hat{Y}(s)$ as input; evaluate the solution at $x = D$ and substitute $\hat{u}(D,s) = C\hat{X}(s)$; take inverse Laplace transform; obtain the delayed versions of $\hat{X}(t)$ and $Y(t)-\hat{Y}(t)$; shift the integration variable to obtain $\int_{t-D}^{t}$.

Second-Order Hyperbolic PDEs:

Wave Equations

Heat equation

$$u_t = u_{xx} \tag{416}$$

Wave equation

$$u_{tt} = u_{xx} \tag{417}$$

Difference roughly analogous to that between

$$\dot{z} + z = 0$$

and

$$\ddot{z} + z = 0$$

Heat equation-type systems have mostly real eigenvalues.

Wave equation-type systems have mostly imaginary eigenvalues.

Classical Boundary Damping/Passive Absorber Control

Vibrating string on a finite interval:

$$
\begin{aligned}
u_{tt} &= u_{xx} && && (418)\\
u_x(0) &= 0 && \text{[free, i.e., force = 0]} && (419)\\
u(1) &= 0 && \text{[pinned]} && (420)
\end{aligned}
$$

Lyapunov function

$$E = \underbrace{\frac{1}{2}\|u_x\|^2}_{\text{potential e.}} + \underbrace{\frac{1}{2}\|u_t\|^2}_{\text{kinetic e.}} \tag{421}$$

$$
\begin{aligned}
\dot{E} &= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t u_{tt} dx && \text{(chain rule)} \\
&= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t u_{xx} dx && \text{(from PDE } u_{tt} = u_{xx}\text{)} \\
&= \int_0^1 u_x u_{xt} dx + \int_0^1 u_t du_x \\
&= \int_0^1 u_x u_{xt} dx + (u_t(x)u_x(x))|_0^1 - \int_0^1 u_{tx} u_x dx && \text{(integration by parts)} \\
&= \cancel{\int_0^1 u_x u_{xt} dx} + (u_t(x)u_x(x))|_0^1 - \cancel{\int_0^1 u_{tx} u_x dx} \\
&= (u_t(x)u_x(x))|_0^1 \\
&= u_t(1)u_x(1) - u_t(0)u_x(0) \\
&= u_x(1)\frac{d}{dt}\underbrace{u(1)}_{=0} \quad - \quad u_t(0)\underbrace{u_x(0)}_{=0} && \text{(from PDE's boundary conditions)} \\
&= 0
\end{aligned}
$$

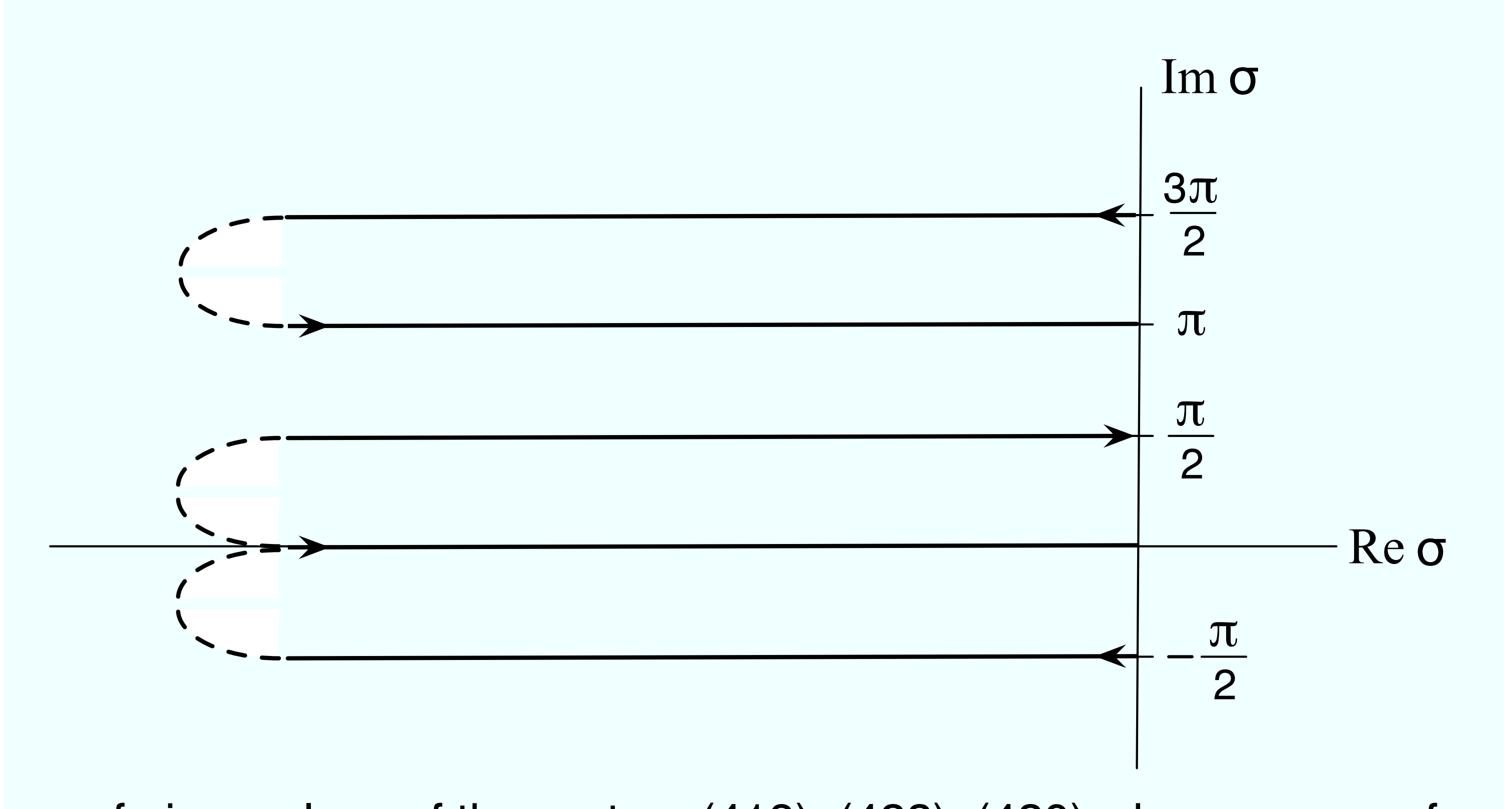
Thus, $E(t) = E(0)$. Energy is conserved. As expected for an *undamped* string.

System is marginally stable.

Classical method of *exponentially* stabilizing this system is by damping on the boundary,

$$\boxed{u_x(0) = c_0 u_t(0)} \qquad c_0 > 0 \tag{422}$$

Force proportional to velocity of free end. Passive damper (absorber).



The locus of eigenvalues of the system (418), (422), (420) when c_0 grows from 0 to 1, and then beyond 1.

Amazing: Damping has effect on the whole domain, though it acts only on the boundary.

How to find the eigenvalues

First, the solution to (418) is sought in the form

$$u(x,t) = e^{\sigma t}\phi(x).$$

Substituting this expression into (418) gives

$$\sigma^2 e^{\sigma t}\phi(x) = e^{\sigma t}\phi''(x),$$

and using the boundary conditions we get

$$\begin{aligned} e^{\sigma t}\phi(1) &= 0 \\ e^{\sigma t}\phi'(0) &= c_0\sigma e^{\sigma t}\phi(0). \end{aligned}$$

We have now arrived at the Sturm-Louiville problem

$$\phi'' - \sigma^2\phi = 0 \tag{423}$$

$$\phi'(0) = c_0\sigma\phi(0) \tag{424}$$

$$\phi(1) = 0. \tag{425}$$

The solution of (423) is given by

$$\phi(x) = e^{\sigma x} + Be^{-\sigma x} \tag{426}$$

From (425) we have $B = -e^{-2\sigma}$, so

$$\phi(x) = e^{\sigma x} - e^{-\sigma(x+2)} \tag{427}$$

From (424)

$$\begin{aligned} \phi'(0) - c_0\sigma\phi(0) &= 0 \quad (428) \\ \sigma(1+e^{2\sigma}) - c_0\sigma(1-e^{2\sigma}) &= 0 \quad (429) \end{aligned}$$

$$\boxed{e^{2\sigma} = -\frac{1-c_0}{1+c_0}} \tag{430}$$

Solving for σ gives

$$\boxed{\boxed{\sigma = -\frac{1}{2}\ln\left|\frac{1+c_0}{1-c_0}\right| + j\pi\begin{cases} n+\frac{1}{2} & 0 \le c_0 < 1 \\ n & c_0 > 1 \end{cases}}} \tag{431}$$

For $c_0 = 1$ the spectrum is at negative infinity and the solution converges to zero in 1 sec.

Caveat:

boundary control (422) on the free end $x = 0$ effective if the string is pinned at $x = 1$.

Backstepping Design:
A String With One Free End and Actuation on the Other End

Consider the wave equation with one free end and force control on the other end:

$$\begin{aligned} u_{tt} &= u_{xx} && (432) \\ u_x(0) &= 0 \qquad \text{[free end, i.e., force = 0]} && (433) \\ u_x(1) &= \text{control force} && (434) \end{aligned}$$

Consider a damping boundary control at the controlled end

$$
\begin{aligned}
u_{tt} &= u_{xx} && (435)\\
u_x(0) &= 0 && (436)\\
u_x(1) &= -c_1 u_t(1), \qquad c_1 > 0 && (437)
\end{aligned}
$$

This system has an arbitrary constant as an equilibrium profile.

A more sophisticated controller at $x = 1$ is needed if the boundary condition at $x = 0$ is to remain free.

We propose the following (backstepping) transformation

$$w(x) = u(x) + c_0 \int_0^x u(y)\,dy \tag{438}$$

which maps the plant into the target system

$$\begin{aligned} w_{tt} &= w_{xx} && (439) \\ w_x(0) &= c_0 w(0) && (440) \\ w_x(1) &= -c_1 w_t(1)\,. && (441) \end{aligned}$$

The idea is that a large c_0 in the boundary condition at $x = 0$ can make $w_x(0) = c_0 w(0)$ behave like $w(0) \approx 0$.

(Large c_0 is not necessary, it is a design option, hence, one should not view this as employing high gain feedback.)

First we analyze the stability of the target system. Consider the Lyapunov function

$$V=\frac{1}{2}\Big(\underbrace{\|w_x\|^2+\|w_t\|^2}+c_0w^2(0)\Big)+\delta\int_0^1(1+x)w_x(x)w_t(x)dx \tag{442}$$

The crucial novelty is the introduction of “indefinite” spatially weighted cross term between w_x and w_t.

Using the Cauchy-Schwartz and Young’s inequalities, one can show that for sufficiently small δ there exist $m_1,\ m_2>0$ such that

$$m_1U\leqslant V\leqslant m_2U\ ,\quad U=\|w_x\|^2+\|w_t\|^2+w^2(0)\,. \tag{443}$$

Therefore V is positive definite.

The derivative of V along the solution of the target system is

$$
\begin{aligned}
\dot{V} &= \int_0^1 w_x w_{tx} dx + \int_0^1 w_t w_{tt} dx + c_0 w(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{tt}) dx \\
&= \int_0^1 w_x w_{tx} dx \\
&\quad + \int_0^1 w_t w_{xx} dx + w_x(0) w_t(0) + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{xx}) dx \\
&= \int_0^1 w_x w_{tx} dx + w_t w_x |_0^1 - \int_0^1 w_t w_{xt} dx + w_x(0) w_t(0) \\
&\quad + \delta \int_0^1 (1+x)(w_{xt} w_t + w_x w_{xx}) dx \\
&= \delta \left(\int_0^1 w_{xt} w_t dx + \int_0^1 w_x w_{xx} dx + \int_0^1 x \underbrace{w_{xt} w_t dx} + \int_0^1 x \underbrace{w_x w_{xx} dx} \right) \\
&\quad + w_t(1) w_x(1) \,.
\end{aligned}
$$

In the last two integrals we notice that

$$w_{xt}w_t dx = \frac{d}{dx}\frac{w_t^2}{2}, \quad w_x w_{xx} dx = \frac{d}{dx}\frac{w_x^2}{2} \tag{444}$$

and use integration by parts:

$$\begin{aligned}
\dot{V} &= w_t(1)w_x(1) + \frac{\delta}{2}\left[(1+x)(w_x^2+w_t^2)\right]\Big|_0^1 - \frac{\delta}{2}\left[(\|w_x\|^2 + \|w_t\|^2\right] \\
&= -c_1 w_t^2 + \delta(w_t^2(1) + w_x^2(1)) - \frac{\delta}{2}\left[w_x^2(0) + w_t^2(0)\right] - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right] \\
&= -\left(c_1 - \delta(1+c_1^2)\right) w_t^2(1) - \frac{\delta}{2}\left(w_t^2(0) + c_0^2 w^2(0)\right) - \frac{\delta}{2}\left[\|w_x\|^2 + \|w_t\|^2\right]
\end{aligned}$$

which is negative definite for

$$\boxed{\delta < \frac{c_1}{1+c_1^2}} \qquad (\delta \text{ is just an analysis parameter})$$

It now follows from (442) and (443) that

$$U(t) \leq M e^{-t/M} U(0)$$

for some possibly large M, which proves exponential stability of the target system.

So, the target system is stable, but what is the control law?

Recall the boundary condition at $x = 1$,

$$w_x(1) = -c_1 w_t(1),$$

and the backstepping transformation,

$$w(x) = u(x) + c_0 \int_0^x u(y)\, dy. \tag{445}$$

The resulting Neumann backstepping controller is obtained by differentiating the transformation (445) and setting $x = 1$:

$$\boxed{u_x(1) = -c_1 u_t(1) - c_0 u(1) - c_1 c_0 \int_0^1 u_t(y) dy} \tag{446}$$

For the best performance (with respect to the case with “pinned” uncontrolled boundary condition in Section), one should choose c_0 large and c_1 around 1.

The corresponding output feedback controller with only boundary sensing is

$$u_x(1) = -c_0\hat{u}(1) - c_1\hat{u}_t(1) + c_0c_1\int_0^1 \hat{u}_t(y)dy. \tag{447}$$

where the observer state is governed by

$$\begin{aligned}
\hat{u}_{tt} &= \hat{u}_{xx} && (448)\\
\hat{u}_x(0) &= \tilde{c}_0(\hat{u}_t(0) - u_t(0)) && (449)\\
\hat{u}(1) &= u(1) && (450)
\end{aligned}$$

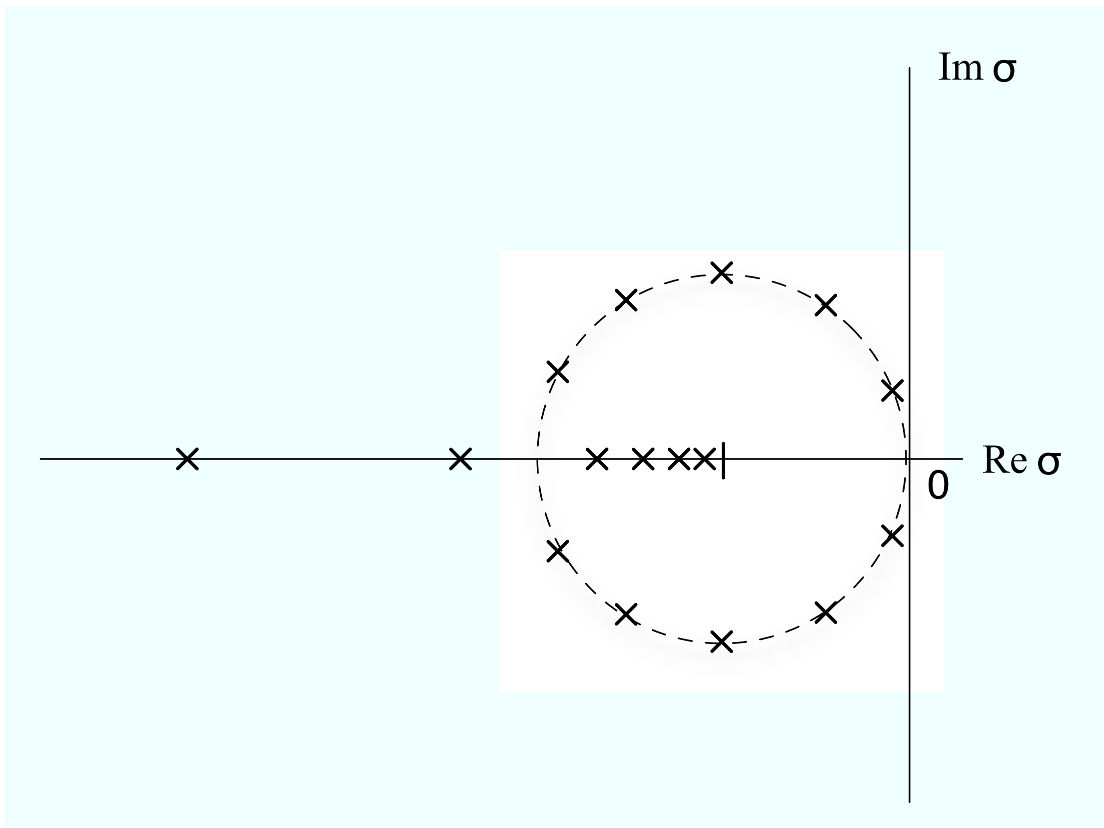
with $\tilde{c}_0 > 0$.

(Why this observer?)

Wave Equation With Kelvin-Voigt Damping

Internal material damping, present in all realistic materials.

$$
\begin{aligned}
u_{tt} &= u_{xx} + \boxed{d u_{xxt}} \qquad d > 0, \text{ small} && (451)\\
u_x(0) &= 0 && (452)\\
u(1) &= \text{control} && (453)
\end{aligned}
$$



Open-loop eigenvalues.

Backstepping transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy$$

Target system

$$w_{tt} = (1 + d\partial_t)\,(w_{xx} - \boxed{cw}) \tag{454}$$

$$w_x(0) = 0 \tag{455}$$

$$w(1) = 0 \tag{456}$$

The n-th pair of eigenvalues σ_n of this system satisfies the quadratic equation

$$\sigma_n^2 + d\left[c + \left(\frac{\pi}{2} + \pi n\right)^2\right]\sigma_n + \left[c + \left(\frac{\pi}{2} + \pi n\right)^2\right] = 0, \tag{457}$$

where $n = 0, 1, 2, \ldots$.

There are two sets of eigenvalues:

- for lower n the eigenvalues reside on the circle

$$\left(\mathrm{Re}(\sigma_n) + \frac{1}{d}\right)^2 + (\mathrm{Im}(\sigma_n))^2 = \frac{1}{d^2} \tag{458}$$

- for higher n the eigenvalues are real, with one branch accumulating towards $-1/d$ as $n \to \infty$ and the other branch converging to $-\infty$

Increasing c moves the eigenvalues along the circle in the negative real direction and decreases the number of them on the circle.

With a very high value of c, all of the eigenvalues can be made real.

(While possible, this would not necessarily be a good idea, neither for transient response, nor for disturbance attenuation, and certainly not from the point of view of control effort.)

The kernel PDE can be shown to be

$$\begin{aligned} k_{xx} &= k_{yy} + ck && (459) \\ k_y(x,0) &= 0 && (460) \\ k(x,x) &= \frac{c}{2}x, && (461) \end{aligned}$$

Note that this is the same PDE as for the reaction-diffusion equation with a Neumann boundary condition on the uncontrolled end.

The controller is given by

$$u(1) = -\int_0^1 c\frac{I_1\left(\sqrt{c(1-y^2)}\right)}{\sqrt{c(1-y^2)}}u(y)\,dy\,. \tag{462}$$

$\boxed{\text{This controller does not depend on } d.}$

Notes

In the 1980s it was shown that feedback laws for the *undamped* wave equation have no robustness margin to time delay (induced by measurement, computation, and actuation).

Homework

1. Show that the transformation

$$w(x) = u(x) + c_0 \int_0^x u(y)\,dy$$

and the boundary control

$$u_x(1) = -c_0 u(1) - c_1 \left(u_t(1) + c_0 \int_0^1 u_t(y)\,dy \right)$$

convert the plant

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(0) &= 0 \end{aligned}$$

into the asymptotically stable system

$$\begin{aligned} w_{tt} &= w_{xx} \\ w_x(0) &= c_0 w(0) \\ w_x(1) &= -c_1 w_t(1)\,. \end{aligned}$$

2. In Exercise 1 determine c_0 and c_1 such that the first pair of poles is around $-1.15 \pm j1.5$. In order to do this, use the GUI available on the class website.

3. Consider the wave equation

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(0) &= -qu(0)\,, \end{aligned}$$

which is unstable with $u(1) = 0$ when $q \geq 1$. Show that the change of variable

$$w(x) = u(x) + (c_0 + q) \int_0^x \mathrm{e}^{q(x-y)} u(y)\, dy$$

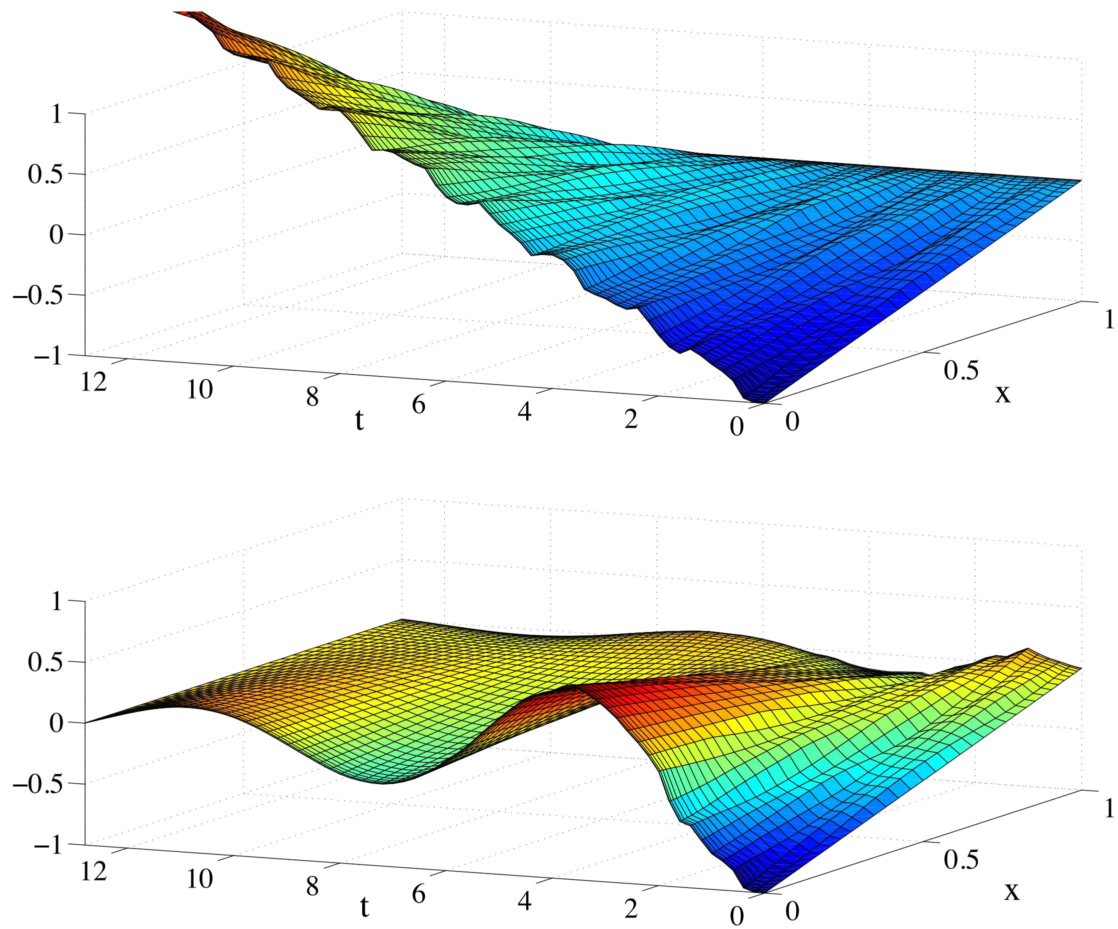
and the boundary feedback

$$u_x(1) = -c_1 u_t(1) - (c_0 + q) u(1) - (c_0 + q) \int_0^1 \mathrm{e}^{q(1-y)} [c_1 u_t(y) + q u(y)] dy$$

convert the closed-loop system into

$$\begin{aligned} w_{tt} &= w_{xx} \\ w_x(0) &= c_0 w(0) \\ w_x(1) &= -c_1 w_t(1)\,. \end{aligned}$$

Nota bene: In figure below it is illustrated that the open-loop plant is unstable and that the feedback controller designed in this exercise successfully stabilizes the plant.



Open-loop (top) and closed-loop (bottom) response of the unstable wave equation.

Beam Equations

Wave (string):

$$u_{tt} - u_{xx} = 0 \tag{463}$$
$$u_x(0) = 0 \quad \text{(free end)} \tag{464}$$
$$u(1) = 0 \quad \text{(pinned end)} \tag{465}$$

Euler-Bernoulli beam:

$$u_{tt} + u_{xxxx} = 0 \tag{466}$$
$$u_{xx}(0) = u_{xxx}(0) = 0 \quad \text{(free end)} \tag{467}$$
$$u(0) = u_x(0) = 0 \quad \text{(clamped end)} \tag{468}$$

Because of the number of spatial derivatives, the wave equation requires one boundary condition per end point, whereas the Euler-Bernoulli beam requires two boundary conditions per end point.

Eigenvalues

Both the beam and the string models have all of their eigenvalues on the imaginary axis. However, while the string eigenvalues are equidistant (growing linearly in n), the beam eigenvalues get further apart as they go up the $j\omega$ axis (they grow quadratically in n).

Is a beam is more difficult to control than a string?

Not necessarily.

Controllability results for beams are valid on arbitrarily short time intervals, whereas for strings they hold only over time intervals that are lower bounded in proportion to the "wave propagation speed" of the string (which physically corresponds to 'string tension').

Slender-beam models:

(1) **Euler-Bernoulli** (neither shear deformations nor rotary inertia)

(2) **Rayleigh**

(3) **shear beam**

(4) **Timoshenko** (both shear deformations and rotary inertia)

(All of the models include the effects of lateral displacement and bending moment, the former contributing the kinetic energy and the latter the strain/potential energy.)

Rayleigh and shear beam models are mathematically identical (include term of the form u_{xxtt}) although they are different physically (the parameters that appear in the two models are different).

Passive damper for Euler-Bernoulli beam

Rotary damper (moment vs. angular velocity):

$$\boxed{u_{xx}(0) = c_0 u_{xt}(0)} \qquad c_0 > 0 \tag{469}$$

This design, while damping the higher modes, is not capable of adding a lot of damping to the first few modes.

Backstepping Control for Euler-Bernoulli Beam

$$u_{tt} + u_{xxxx} = 0 \tag{470}$$

with “sliding end” boundary conditions

$$u_x(0) = u_{xxx}(0) = 0 \tag{471}$$

Control boundary conditions at $x = 1$:

$$u(1) = U_1, \quad u_{xx}(1) = U_2 \tag{472}$$

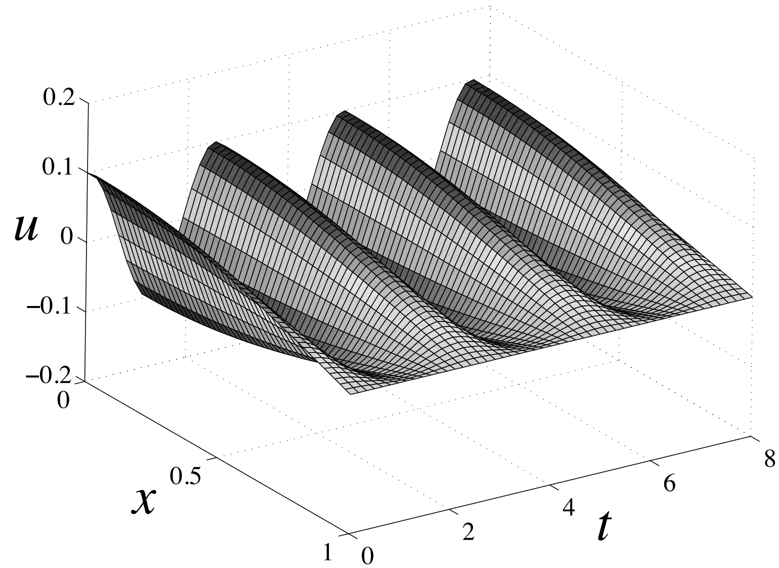
Introduce a new complex variable

$$v = u_t - ju_{xx} \tag{473}$$

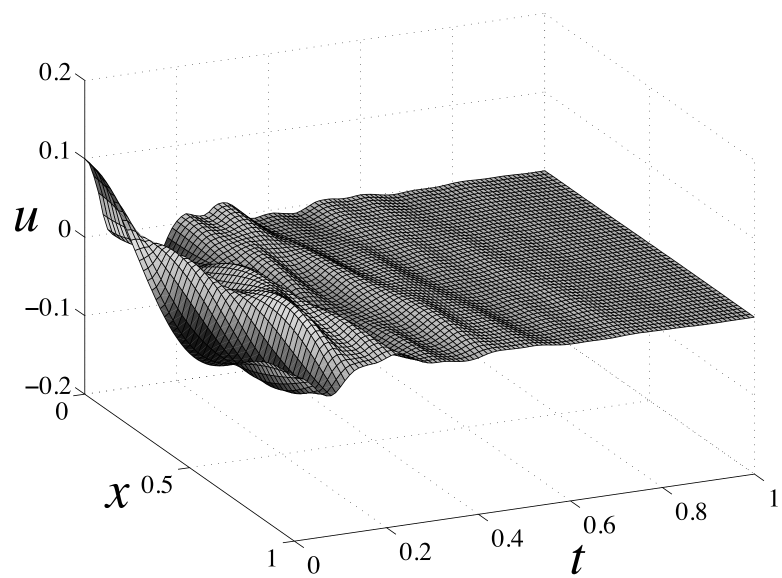
to convert the EB beam into the Schrödinger equation

$$v_t = -jv_{xx}$$

Parabolic-like PDE with an imaginary diffusion coefficient.



Open-loop response of the Euler-Bernoulli beam.



Closed-loop response with backstepping controller (full-state fbk of u and u_t).

Shear Beam

$$u_{tt} - \varepsilon u_{xxtt} + u_{xxxx} = 0 \tag{474}$$

ε = small constant inversely proportional to the shear modulus (and unrelated to damping)

Write in the form of the wave equation coupled with a second order ODE in x:

$$\varepsilon u_{tt} = u_{xx} - \alpha_x \tag{475}$$

$$0 = \varepsilon \alpha_{xx} - \alpha + u_x \tag{476}$$

α = deflection angle due to the bending of the beam

Free-end boundary condition:

$$u_x(0) = \alpha(0) \tag{477}$$

$$\alpha_x(0) = 0 \tag{478}$$

One can verify that this model is equivalent to the model (474) following these steps:

a) $(475)_x + (476) = (\star)$

b) $(\star)_x = (\star\star)$

c) $(\star\star) - \frac{1}{\varepsilon}(475) = (474)$

Backstepping Control for Shear Beam

The first step of our design is to solve the ODE (476) for α:

$$\alpha(x) = \cosh(bx)\alpha(0) - b\int_0^x \sinh(b(x-y))u_y(y)dy \tag{479}$$

where

$$b = 1/\sqrt{\varepsilon}$$

This solution is easily obtained via Laplace transform in the spatial variable x.

The constant $\alpha(0)$ in (479) can be expressed in terms of $\alpha(1)$ in the following way:

$$\begin{aligned}\alpha(0) &= \frac{1}{\cosh(b)}\left[\alpha(1) + b\int_0^1 \sinh(b(1-y))u_y(y)dy\right]\\ &= \frac{1}{\cosh(b)}\left[\alpha(1) + b\sinh(b(1-y))u(y)|_0^1 + b^2\int_0^1 \cosh(b(1-y))u(y)dy\right]\\ &= \frac{1}{\cosh(b)}\left[\alpha(1) - b\sinh(b)u(0) + b^2\int_0^1 \cosh(b(1-y))u(y)dy\right].\end{aligned} \tag{480}$$

The integral term on the right hand side of this equality is not spatially causal, due to the fourth derivative in the original shear beam model (474). To put the system into a strict-feedback form, we eliminate this integral by choosing

$$\boxed{\alpha(1) = b\sinh(b)u(0) - b^2\int_0^1 \cosh(b(1-y))u(y)dy} \tag{481}$$

so that $\alpha(0) = 0$. Then we have

$$\alpha(x) = b\sinh(bx)u(0) - b^2\int_0^x \cosh(b(x-y))u(y)dy\,. \tag{482}$$

Differentiating $\alpha(x)$ with respect to x and substituting the result into the wave equation (475) we get the system in the form ready for the control design:

$$\varepsilon u_{tt} = u_{xx} + b^2u - b^2\cosh(bx)u(0) + b^3\int_0^x \sinh(b(x-y))u(y)dy. \tag{483}$$

Following our procedure, we use the transformation

$$w(x) = u(x) - \int_0^x k(x,y)u(y)\,dy \tag{484}$$

to map the system (483) into the following exponentially stable target system

$$\varepsilon w_{tt} = w_{xx} \tag{485}$$

$$w_x(0) = c_0 w(0) \tag{486}$$

$$w_x(1) = -c_1 w_t(1)\,, \tag{487}$$

where c_0 and c_1 are design parameters. Perhaps somewhat surprisingly, we use the same target system here as in the control design for the wave equation.

Substituting the transformation (484) into the target system, one can derive the following PDE for the kernel $k(x,y)$:

$$k_{xx} = k_{yy} + b^2 k - b^3 \sinh(b(x-y)) + b^3 \int_y^x k(x,\xi)\sinh(b(\xi - y))d\xi \tag{488}$$

$$k(x,x) = -\frac{b^2}{2}x - c_0 \tag{489}$$

$$k_y(x,0) = b^2 \int_0^x k(x,y)\cosh(by)dy - b^2\cosh(bx)\,. \tag{490}$$

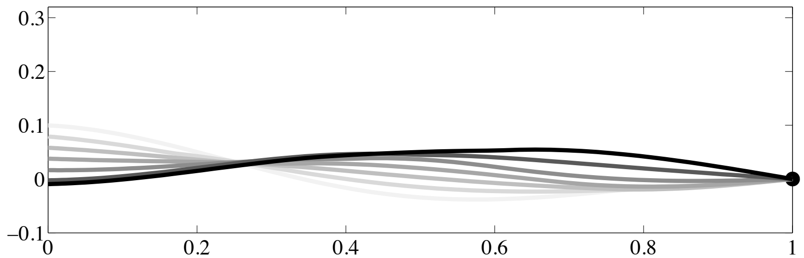
This PDE has to be solved numerically.

The second boundary controller (the first one is given by (481)) is obtained by differentiating (484) with respect to x and setting $x = 1$:

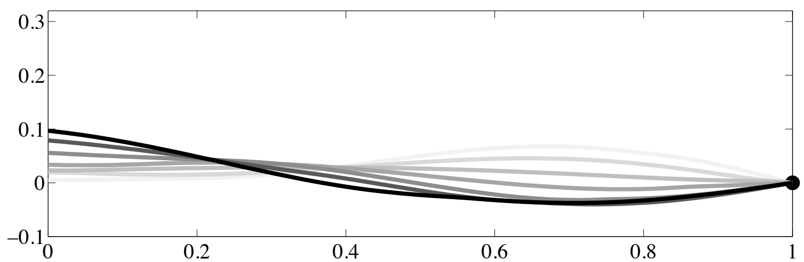
$$\boxed{u_x(1) = k(1,1)u(1) + \int_0^1 k_x(1,y)u(y)dy - c_1 u_t(1) + c_1 \int_0^1 k(1,y)u_t(y)dy} \qquad (491)$$

In a similar way one can design the observer to avoid the need of position and velocity measurements along the beam.

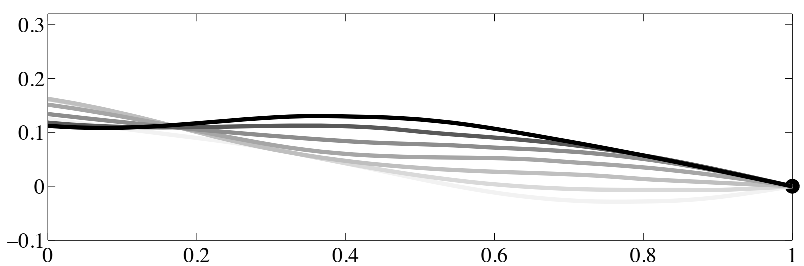
Uncontrolled Beam



$t \in [0, 0.4]$

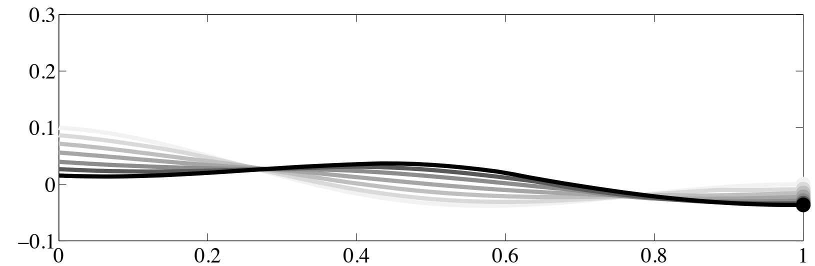


$t \in [0.4, 1.5]$

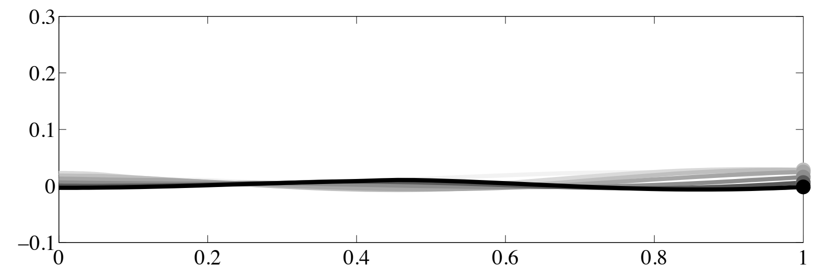


$t \in [1.5, 2.1]$

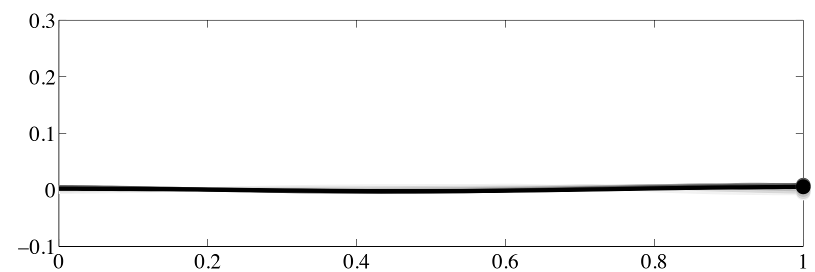
Controlled Beam



$t \in [0, 0.4]$



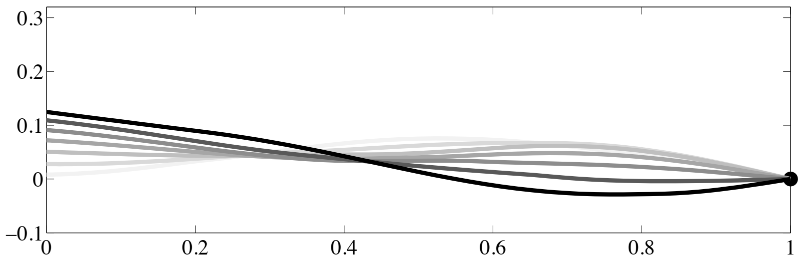
$t \in [0.4, 1.5]$



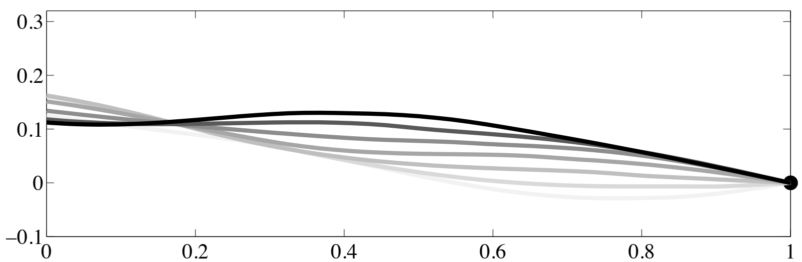
$t \in [1.5, 2.1]$

Snapshots of the shear beam movements with
increasing darkness denoting increasing time in the sequences.

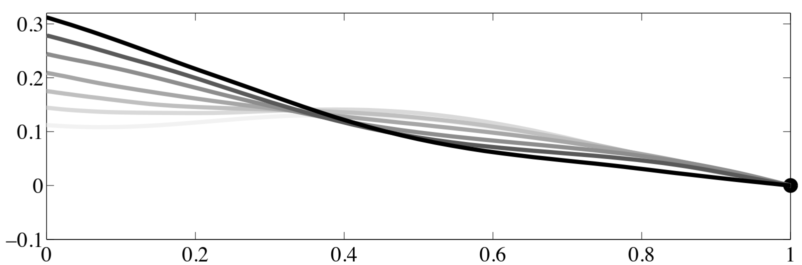
Uncontrolled Beam



$t \in [2.1, 2.8]$

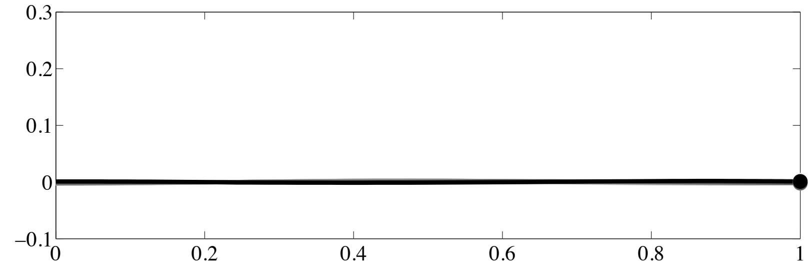


$t \in [2.8, 3.6]$

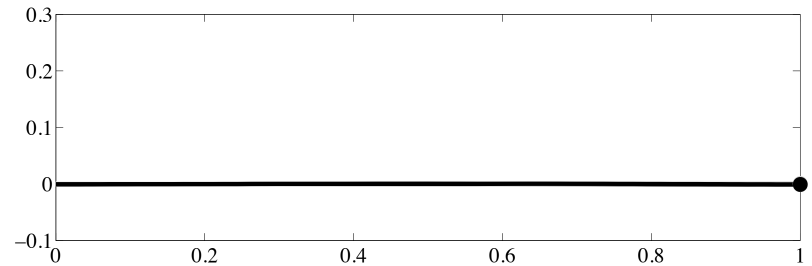


$t \in [3.6, 4.2]$

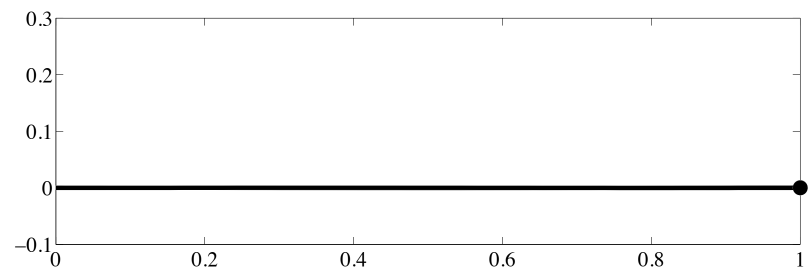
Controlled Beam



$t \in [2.1, 2.8]$

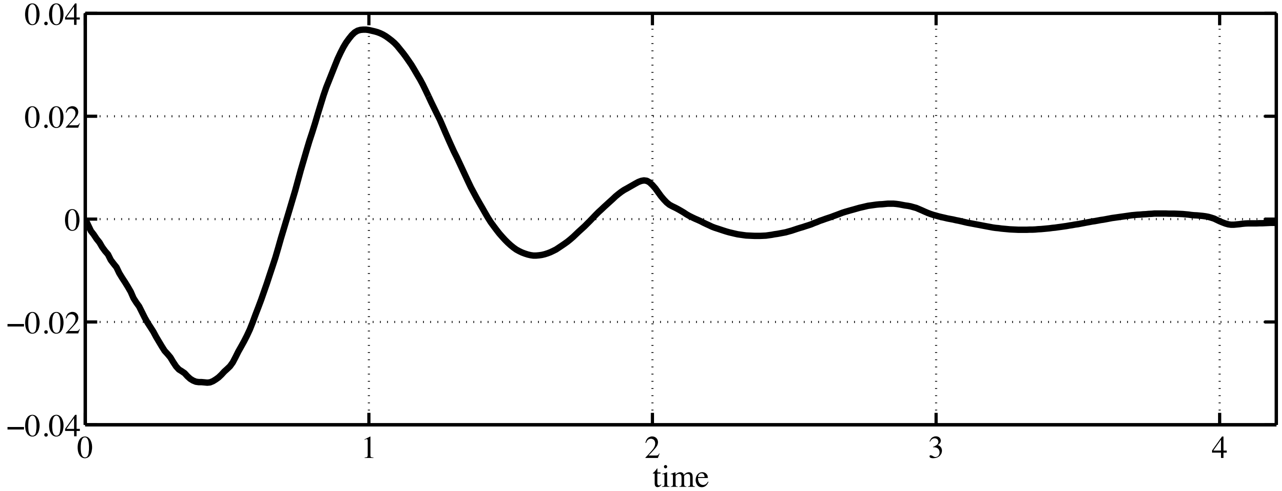


$t \in [2.8, 3.6]$

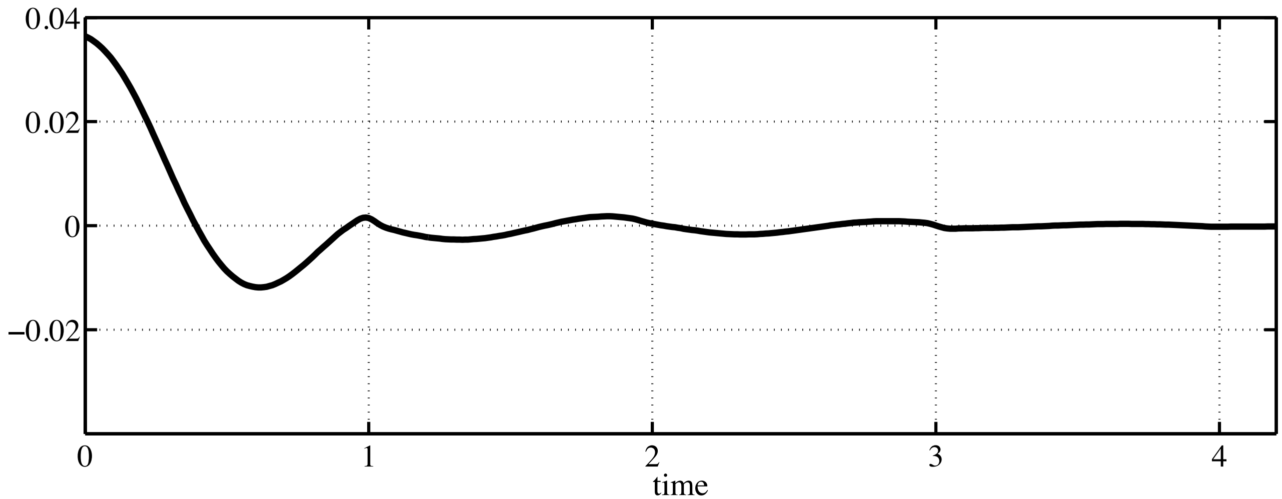


$t \in [3.6, 4.2]$

Control $u(1,t)$



Control $\alpha(1,t)$



The controls for the shear beam

Timoshenko Beam

$$\begin{aligned} \varepsilon u_{tt} &= (1+d\partial_t)(u_{xx}-\theta_x) & (492) \\ \mu\varepsilon\theta_{tt} &= (1+d\partial_t)(\varepsilon\theta_{xx}+a(u_x-\theta)) & (493) \end{aligned}$$

$u(x,t)$ = displacement
$\theta(x,t)$ = deflection angle

$\varepsilon \propto$ 1/(shear modulus)
$\mu \propto$ moment of inertia
$a \propto$ cross-sectional area
d = Kelvin-Voigt damping

Free-end boundary condition:

$$\begin{aligned} u_x(0,t) &= \theta(0,t) && \text{(zero force applied at the tip)} & (494) \\ \theta_x(0,t) &= 0 && \text{(zero moment applied at the tip)} & (495) \end{aligned}$$

Shear beam is a singular perturbation of the Timoshenko beam as $\mu \to 0$.

Unlike the Euler-Bernoulli, shear, and Rayleigh beam models, which are second order in t, the Timoshenko beam is fourth order in t.

Motion Planning for PDEs

- Start from an output reference trajectory, for example, $u^r(0,t)$, as the desired temporal waveform of the system output $u(0,t)$ ar $x = 0$.

- Generate the state trajectory $u^r(x,t)$ for all x (including $x = 1$, which produces the *control reference* $u^r(1,t)$ in the case of Direchlet actuation).

- Combine this result with a feedback control law for the tracking error system $u - u^r$,

$$\boxed{u(1,t)} - u^r(1,t) = \int_0^1 k(1,y)(u(y,t) - u^r(y,t))\,dy$$

to stabilize the trajectory $u^r(x,t)$, and to force the output $u(0,t)$ to track the output trajectory $u^r(0,t)$.

We forego generality and focus on examples with reference outputs common in practice—exponential, polynomial, and sinusoidal signals.

“Dirichlet” outputs $u(0,t)$ and the “Neumann” outputs $u_x(0,t)$.

Examples for heat equation, reaction-diffusion equation, wave equation, Euler-Bernoulli beam, and a first order hyperbolic PDE.

Backstepping allows us to solve the tracking problem for the original plant by using the motion planning solution for the simpler, target system:

$$
\begin{aligned}
u(1,t) &= \underbrace{u^r(1,t)} + \int_0^1 k(1,y)(u(y,t) - \underbrace{u^r(y,t)})\,dy \\
&= \underbrace{u^r(1,t) - \int_0^1 k(1,y)u^r(y,t)\,dy} + \int_0^1 k(1,y)u(y,t)\,dy \\
&= w^r(1,t) + \int_0^1 k(1,y)u(y,t)\,dy
\end{aligned}
$$

Trajectory Generation

Example 7 (Heat eqn and Dirichlet output, anti-collocated w/ input)

$$
\begin{aligned}
u_t &= u_{xx} && (496)\\
u_x(0) &= 0 && (497)
\end{aligned}
$$

Reference output

$$u^r(0,t) = 1 + 2t - t^2 . \qquad (498)$$

Find the reference input $u^r(1,t)$. As an intermediate step, construct the full state trajectory $u^r(x,t)$ which simultaneously satisfies (496), (497), and (498).

Search for the state trajectory in the following form:

$$\boxed{u^r(x,t) = \sum_{k=0}^{\infty} a_k(t) \frac{x^k}{k!}} \qquad (499)$$

Taylor series in x with time varying coefficients $a_k(t)$ that need to be determined from (496)–(498).

From (498)–(499) we see that

$$a_0(t) = u^r(0,t) = 1 + 2t - t^2. \tag{500}$$

The boundary condition (497) gives

$$a_1(t) = u^r_x(0,t) = 0\,. \tag{501}$$

The next step is to substitute (499) into (496):

$$\begin{aligned}
\sum_{k=0}^{\infty} \dot{a}_k(t)\frac{x^k}{k!} &= \frac{\partial^2}{\partial x^2}\sum_{k=0}^{\infty} a_k(t)\frac{x^k}{k!} \\
&= \sum_{k=2}^{\infty} a_k(t)\frac{k(k-1)x^{k-2}}{k!} \\
&= \sum_{k=2}^{\infty} a_k(t)\frac{x^{k-2}}{(k-2)!} \\
&= \sum_{k=0}^{\infty} a_{k+2}(t)\frac{x^k}{k!}.
\end{aligned} \tag{502}$$

We get the recursive relationship

$$\boxed{a_{k+2}(t) = \dot{a}_k(t)} \tag{503}$$

Using (501) and (500) with (503) results in

$$\begin{array}{rclll}
a_0 & = & 1+2t-t^2, & a_1 = 0 & \\
a_2 & = & 2-2t, & a_3 = 0 & \\
a_4 & = & -2, & a_5 = 0 & \\
a_6 & = & 0, & a_i = 0 & \text{for } i > 6.
\end{array}$$

This gives the reference state trajectory

$$u^r(x,t) = 1+2t+t^2+(1-t)x^2-\frac{1}{12}x^4,$$

and the input signal

$$u^r(1,t) = \frac{23}{12}+t-t^2.$$

Note that the output matches reference output trajectory only in the extremely unlikely case that the initial condition of the plant is satisfied by the state trajectory, that is, if

$$u(x,0) = 1 + x^2 - \frac{1}{12}x^4$$

To asymptotically track the reference signal as $t \to \infty$ for an arbitrary initial condition we need to apply **feedback**.

In the subsequent examples the output reference will not be polynomial and the series will be infinite and has to be summed.

Example 8 (Reaction-diffusion eqn and Dirichlet output, anti-collocated w/ input)

$$u_t = u_{xx} + \lambda u \tag{504}$$

$$u_x(0) = 0 \tag{505}$$

Reference output

$$u^r(0,t) = \mathrm{e}^{\alpha t} \tag{506}$$

α may be real, imaginary, or complex.

Again postulate the full state reference trajectory in the form

$$u^r(x,t) = \sum_{k=0}^{\infty} a_k(t) \frac{x^k}{k!} \tag{507}$$

From (506) and the boundary condition (505) we have

$$a_0(t) = \mathrm{e}^{\alpha t}, \qquad a_1(t) = 0$$

From the PDE (504) we get

$$\boxed{a_{k+2}(t) = \dot{a}_k(t) - \lambda a_k(t)}$$

These conditions give

$$\begin{aligned} a_{2k+1} &= 0 \\ a_{2k+2} &= \dot{a}_{2k} - \lambda a_{2k} \\ a_2 &= (\alpha - \lambda) e^{\alpha t} \\ a_4 &= (\alpha - \lambda)^2 e^{\alpha t} \\ a_{2k} &= (\alpha - \lambda)^k e^{\alpha t} \end{aligned}$$

for $k = 0, 1, 2, \ldots$

The state trajectory is

$$
\begin{aligned}
u^r(x,t) &= \sum_{k=0}^{\infty} (\alpha-\lambda)^k e^{\alpha t} \frac{x^{2k}}{(2k)!} \\
&= \sum_{k=0}^{\infty} e^{\alpha t} \frac{(\sqrt{\alpha-\lambda}x)^{2k}}{(2k)!} \\
&= e^{\alpha t} \begin{cases} \cosh(\sqrt{\alpha-\lambda}x) & \alpha \geq \lambda \\ \cos(\sqrt{\lambda-\alpha}x) & \alpha < \lambda \end{cases}
\end{aligned}
\tag{508}
$$

The reference input is

$$
u^r(1,t) = e^{\alpha t} \begin{cases} \cosh(\sqrt{\alpha-\lambda}) & \alpha \geq \lambda \\ \cos(\sqrt{\lambda-\alpha}) & \alpha < \lambda \end{cases}
$$

The following formulae are useful when calculating the trajectories for sinusoidal reference outputs in upcoming examples:

$$\cosh(a) = \sum_{k=0}^{\infty} \frac{a^{2k}}{(2k)!} \qquad \sinh(a) = \sum_{k=0}^{\infty} \frac{a^{2k+1}}{(2k+1)!}$$

$$\cosh(ja) = \cos(a) \qquad \sinh(ja) = j\sin(a)$$

$$\cos(ja) = \cosh(a) \qquad \sin(ja) = j\sinh(a)$$

Example 9 (Heat eqn and sinusoidal Dirichlet output, anti-collocated w/ input)

$$u_t = u_{xx} \tag{509}$$

$$u_x(0) = 0 \tag{510}$$

Reference output

$$u^r(0,t) = \sin(\omega t) \tag{511}$$

Since

$$\sin(\omega t) = \mathrm{Im}\{e^{j\omega t}\}$$

we can get the reference trajectory by setting $\lambda = 0$ and $\alpha = j\omega$ in the previous example:

$$u^r(x,t) = \mathrm{Im}\left\{\cosh(\sqrt{j\omega}x)e^{j\omega t}\right\} \tag{512}$$

Use the identity

$$\boxed{\sqrt{j} = \frac{1+j}{2}}$$

Using the trig/hyperbolic identities, we get

$$
\begin{aligned}
u^r(x,t) &= \mathrm{Im}\left\{\cosh\left((1+j)\sqrt{\frac{\omega}{2}}x\right)e^{j\omega t}\right\} \\
&= \mathrm{Im}\left\{\frac{e^{\sqrt{\frac{\omega}{2}}x+j(\omega t+\sqrt{\frac{\omega}{2}}x)}+e^{-\sqrt{\frac{\omega}{2}}x+j(\omega t-\sqrt{\frac{\omega}{2}}x)}}{2}\right\} \\
&= \frac{1}{2}e^{\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t+\sqrt{\frac{\omega}{2}}x\right)+\frac{1}{2}e^{-\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t-\sqrt{\frac{\omega}{2}}x\right) \qquad (513)
\end{aligned}
$$

Finally, the reference input is

$$
u^r(1,t) = \frac{1}{2}e^{\sqrt{\frac{\omega}{2}}}\sin\left(\omega t+\sqrt{\frac{\omega}{2}}\right)+\frac{1}{2}e^{-\sqrt{\frac{\omega}{2}}}\sin\left(\omega t-\sqrt{\frac{\omega}{2}}\right)
$$

This signal can rewritten as

$$
\begin{aligned}
u^r(1,t) &= \frac{1}{2}e^{\sqrt{\frac{\omega}{2}}}\sin\left(\omega t+\sqrt{\frac{\omega}{2}}\right)+\frac{1}{2}e^{-\sqrt{\frac{\omega}{2}}}\sin\left(\omega t-\sqrt{\frac{\omega}{2}}\right) \\
&= \frac{1}{2}e^{\sqrt{\frac{\omega}{2}}}\left[\sin(\omega t)\cos\left(\sqrt{\frac{\omega}{2}}\right)+\cos(\omega t)\sin\left(\sqrt{\frac{\omega}{2}}\right)\right] \\
&\quad +\frac{1}{2}e^{-\sqrt{\frac{\omega}{2}}}\left[\sin(\omega t)\cos\left(\sqrt{\frac{\omega}{2}}\right)-\cos(\omega t)\sin\left(\sqrt{\frac{\omega}{2}}\right)\right] \\
&= \sin(\omega t)\cos\left(\sqrt{\frac{\omega}{2}}\right)\cosh\left(\sqrt{\frac{\omega}{2}}\right)+\cos(\omega t)\sin\left(\sqrt{\frac{\omega}{2}}\right)\sinh\left(\sqrt{\frac{\omega}{2}}\right)
\end{aligned}
$$

The final sinusoidal reference input is

$$
\boxed{u^r(1,t)=\underbrace{\sinh\left(\sqrt{\frac{\omega}{2}}\right)\sqrt{1+\cos^2\left(\sqrt{\frac{\omega}{2}}\right)}}_{\text{amplitude}(\omega)}\sin\left(\omega t+\underbrace{\arctan\left(\tan\left(\sqrt{\frac{\omega}{2}}\right)\tanh\left(\sqrt{\frac{\omega}{2}}\right)\right)}_{\text{phase}(\omega)}\right)}
$$

The examples so far have all dealt with Dirichlet type outputs $u(0,t)$.

The next example deals with the output, $u_x(1,t)$, which is Neumann type and **collocated with input!**

Example 10 (Heat eqn and Neumann output, collocated w/ input)

$$\begin{aligned} u_t &= u_{xx} \quad (514) \\ u(0) &= 0 \quad (515) \end{aligned}$$

Reference output

$$u_x^r(1) = \sin(\omega t) \quad (516)$$

Postulating $u^r(x,t)$ in the form (507), we get $\boxed{a_{i+2} = \dot{a}_i}$ and the boundary condition gives

$$\begin{aligned} a_{2k} &= 0 \quad (517) \\ a_{2k+1} &= a_1^{(k)} \quad (518) \end{aligned}$$

The state trajectory becomes

$$u^r(x,t) = \sum_{k=0}^{\infty} a_1^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!} \quad (519)$$

The output reference is

$$u_x^r(1,t) = \sum_{k=0}^{\infty} \frac{a_1^{(k)}(t)}{(2k)!} = \sin(\omega t) = \mathrm{Im}\{e^{j\omega t}\}\,. \tag{520}$$

Suppose that

$$a_1(t) = \mathrm{Im}\{Ae^{j\omega t}\},$$

where A is a constant to be determined. Then

$$a_1^{(k)}(t) = \mathrm{Im}\{A(j\omega)^k e^{j\omega t}\}.$$

From (520) we get

$$\begin{aligned} \mathrm{Im}\left\{Ae^{j\omega t}\sum_{k=0}^{\infty}\frac{(\sqrt{j\omega})^{2k}}{(2k)!}\right\} &= \mathrm{Im}\{e^{j\omega t}\} && (521)\\ Ae^{j\omega t}\cosh(\sqrt{j\omega}) &= e^{j\omega t}, && (522) \end{aligned}$$

so that

$$\boxed{A = \frac{1}{\cosh(\sqrt{j\omega})}}$$

The state trajectory is now

$$
\begin{aligned}
u^r(x,t) &= \sum_{k=0}^{\infty} a_1^{(k)}(t)\frac{x^{2k+1}}{(2k+1)!} && (523)\\
&= \text{Im}\left\{A\sum_{k=0}^{\infty}(j\omega)^k\frac{x^{2k+1}}{(2k+1)!}e^{j\omega t}\right\} && (524)\\
&= \text{Im}\left\{\frac{A}{\sqrt{j\omega}}\sum_{k=0}^{\infty}\frac{(\sqrt{j\omega}x)^{2k+1}}{(2k+1)!}e^{j\omega t}\right\} && (525)\\
&= \text{Im}\left\{\frac{\sinh(\sqrt{j\omega}x)}{\sqrt{j\omega}\cosh(\sqrt{j\omega})}e^{j\omega t}\right\}, && (526)
\end{aligned}
$$

which gives the reference input

$$
\boxed{u^r(1,t) = \text{Im}\left\{\frac{\tanh(\sqrt{j\omega})}{\sqrt{j\omega}}e^{j\omega t}\right\}} \qquad (527)
$$

All of the examples so far were for parabolic PDEs. The remaining examples are for hyperbolic PDEs.

Example 11 (Wave eqn and Dirichlet output, anti-collocated w/ input)

$$
\begin{aligned}
u_{tt} &= u_{xx} && (528)\\
u(0) &= 0 && (529)
\end{aligned}
$$

Reference output

$$
u_x^r(0,t) = \sin(\omega t) \tag{530}
$$

Searching for $u^r(x,t)$ in the form (507), we get

$$
\begin{aligned}
a_0 &= 0, \;\; a_1(t) = \sin(\omega t) = \mathrm{Im}\{e^{j\omega t}\}\\
a_{i+2} &= \ddot{a}_i(t),
\end{aligned}
$$

which gives

$$
\begin{aligned}
a_{2k} &= 0\\
a_{2k+1}(t) &= (j\omega)^{2k}\mathrm{Im}\{e^{j\omega t}\}.
\end{aligned}
$$

The state reference becomes

$$\begin{aligned} u^r(x,t) &= \operatorname{Im}\left\{ \frac{e^{j\omega t}}{j\omega} \sum_{k=0}^{\infty} \frac{(j\omega x)^{2k+1}}{(2k+1)!} \right\} \\ &= \operatorname{Im}\left\{ \frac{e^{j\omega t}}{j\omega} \sinh(j\omega x) \right\} \\ &= \operatorname{Im}\left\{ \frac{e^{j\omega t}}{\omega} \sin(\omega x) \right\} \\ &= \frac{1}{\omega} \sin(\omega x) \sin(\omega t), \end{aligned}$$

and the reference input is

$$\boxed{u^r(1,t) = \frac{\sin(\omega)}{\omega} \sin(\omega t)} \tag{531}$$

Note that for the same desired reference output trajectory, the reference input for the heat equation (Example 9) has a much more complicated form.

Example 12 (Wave equation with Kelvin-Voigt damping)

$$
\begin{aligned}
\varepsilon u_{tt} &= (1+d\partial_t)u_{xx} &\quad (532)\\
u_x(0) &= 0 &\quad (533)
\end{aligned}
$$

Reference output

$$u^r(0,t) = \sin(\omega t) \tag{534}$$

Reference state trajectory:

$$
\begin{aligned}
u^r(x,t) = \frac{1}{2}\Bigg[& e^{\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2 d^2}-1}}{\sqrt{2}\sqrt{1+\omega^2 d^2}}x} \sin\left(\omega\left(t+\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2 d^2}+1}}{\sqrt{2}\sqrt{1+\omega^2 d^2}}x\right)\right) \\
& + e^{-\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2 d^2}-1}}{\sqrt{2}\sqrt{1+\omega^2 d^2}}x} \sin\left(\omega\left(t-\sqrt{\varepsilon}\frac{\omega\sqrt{\sqrt{1+\omega^2 d^2}+1}}{\sqrt{2}\sqrt{1+\omega^2 d^2}}x\right)\right)\Bigg].
\end{aligned} \tag{535}
$$

Example 13 (Euler-Bernoulli beam w/ two outputs and two anti-collocated inputs)

$$
\begin{aligned}
u_{tt} + u_{xxxx} &= 0 && (536)\\
u_{xx}(0) &= u_{xxx}(0) = 0 && (537)
\end{aligned}
$$

Reference outputs

$$
\begin{aligned}
u^r(0,t) &= \sin(\omega t) && (538)\\
u^r_x(0,t) &= 0 && (539)
\end{aligned}
$$

Note that, because the beam equation is fourth order in x, we are free to assign two independent reference outputs, and free to choose two reference inputs.

Searching for $u^r(x,t)$ in the form (507), we get

$$
\begin{aligned}
a_0 &= \sin(\omega t) = \mathrm{Im}\left\{e^{j\omega t}\right\}, && (540)\\
a_1 &= a_2 = a_3 = 0, && (541)\\
a_{i+4} &= -\ddot{a}_i\,. && (542)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_{4k} &= (-1)^k a_0^{(2k)} = \mathrm{Im}\left\{(-1)^k (j\omega)^{2k} e^{j\omega t}\right\} = \omega^{2k}\sin(\omega t) && (543)\\
a_{4k+1} &= a_{4k+2} = a_{4k+3} = 0\,. && (544)
\end{aligned}
$$

The reference trajectory becomes

$$
\begin{aligned}
u^r(x,t) &= \sum_{k=0}^{\infty} \omega^{2k} \frac{x^{4k}}{(4k)!} \sin(\omega t) \\
&= \sum_{k=0}^{\infty} \frac{(\sqrt{\omega} x)^{4k}}{(4k)!} \sin(\omega t) \\
&= \frac{1}{2}[\cosh(\sqrt{\omega} x) + \cos(\sqrt{\omega} x)] \sin(\omega t)\,,
\end{aligned}
$$

and reference inputs are

$$
\begin{aligned}
u^r(1,t) &= \frac{1}{2}[\cosh(\sqrt{\omega}) + \cos(\sqrt{\omega})] \sin(\omega t)\,, \\
u^r_x(1,t) &= \frac{\sqrt{\omega}}{2}[\sinh(\sqrt{\omega} x) - \sin(\sqrt{\omega} x)] \sin(\omega t)\,.
\end{aligned}
$$

Example 14 (First order hyperbolic PDE)

$$u_t \quad = \quad u_x + g u(0) \tag{545}$$

Reference output

$$u^r(0,t) = \sin(\omega t) \tag{546}$$

Searching for the reference trajectory in the form (507), we get

$$\begin{aligned} a_0 &= \sin(\omega t) = \mathrm{Im}\left\{e^{j\omega t}\right\}, \\ a_1 &= \dot{a}_0 - g u(0) = \mathrm{Im}\left\{(j\omega - g)e^{j\omega t}\right\}, \\ a_{k+1} &= \dot{a}_k = \mathrm{Im}\left\{(j\omega - g)(j\omega)^k e^{j\omega t}\right\} \\ &= \mathrm{Im}\left\{\left(1 - \frac{g}{j\omega}\right)(j\omega)^{k+1} e^{j\omega t}\right\}. \end{aligned}$$

The reference trajectory becomes

$$
\begin{aligned}
u^r(x,t) &= \mathrm{Im}\left\{ e^{j\omega t} + \left(1 - \frac{g}{j\omega}\right) \sum_{k=1}^{\infty} \frac{(j\omega x)^k}{k!} e^{j\omega t} \right\} \\
&= \mathrm{Im}\left\{ \left[\frac{g}{j\omega} + \left(1 - \frac{g}{j\omega}\right) e^{j\omega x} \right] e^{j\omega t} \right\} \\
&= -\frac{g}{\omega}[\cos(\omega t) - \cos(\omega(t+x))] + \sin(\omega(t+x))\,.
\end{aligned}
$$

which gives the reference input

$$
\boxed{u^r(1,t) = \frac{g}{\omega}[\cos(\omega(t+1)) - \cos(\omega t)] + \sin(\omega(t+1))} \tag{547}
$$

Trajectory Tracking

$$u_t = u_x + gu(0)$$

Recall that we established for any plant that the control law for tracking is

$$u(1,t) \quad = \quad w^r(1,t) + \int_0^1 k(1,y)u(y,t)\,dy$$

For (548) the target system is

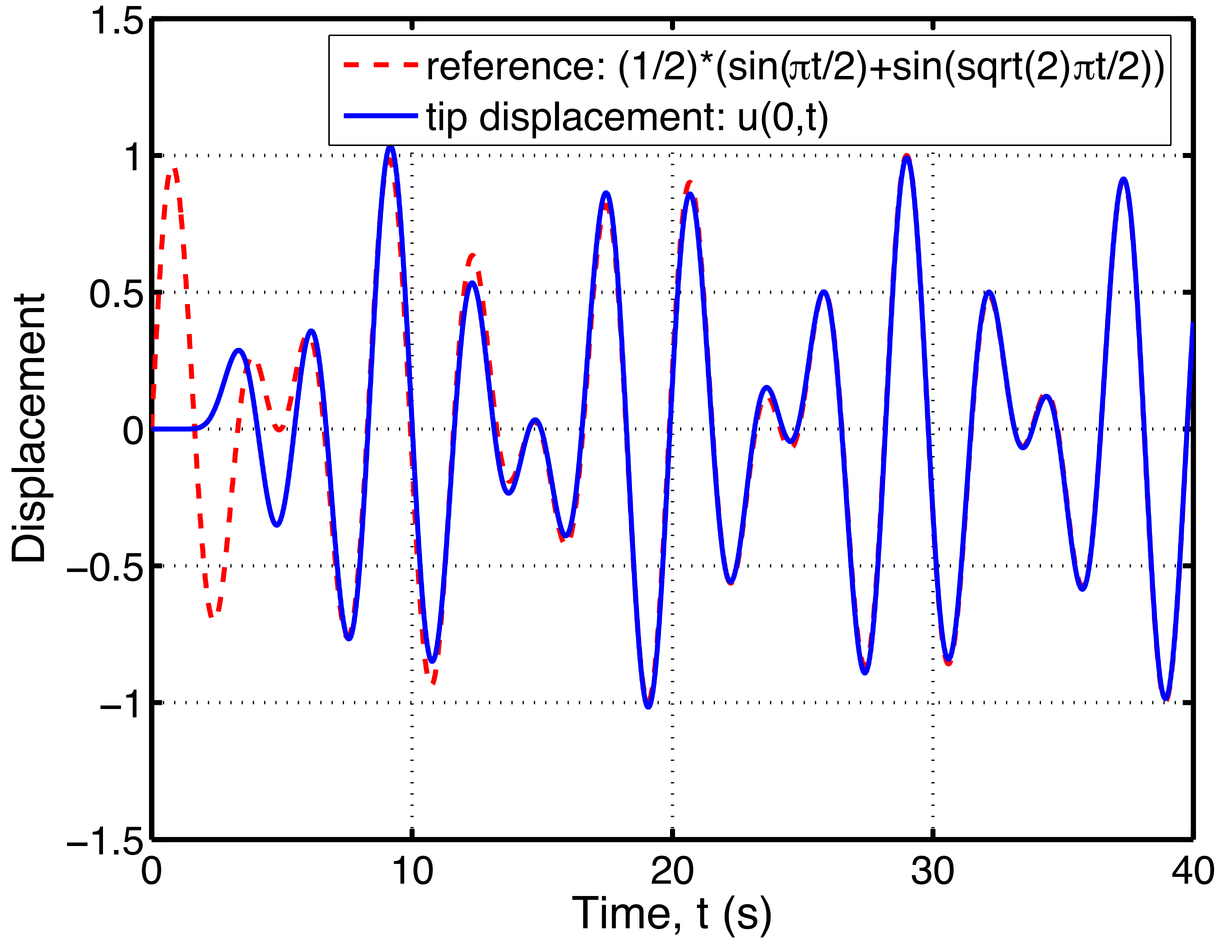
$$w_t \quad = \quad w_x \tag{548}$$

Since $w(0,t) = u(0,t)$ and $u^r(0,t) = \sin(\omega t)$, we have that

$$w^r(1,t) = \sin(\omega(t+1))$$

Hence, we obtain the complete control law (feedforward plus feedback) as

$$\boxed{u(1,t) = \sin(\omega(t+1)) - \int_0^1 \underbrace{ge^{g(1-y)}}_{-k(1,y)} u(y,t)dy} \tag{549}$$



Asymptotic tracking for the wave equation with Kelvin-Voigt damping in Example 12, for the *almost periodic* trajectory given by

$$u^r(0,t) = \frac{1}{2}\left[\sin\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\sqrt{2}\pi}{2}t\right)\right]. \tag{550}$$

Notes and References

The foundations for motion planning for several classes of PDEs were laid in the late 1990's in a series of papers by Rouchon, Martin, Fliess, Petit, and coworkers. These results extended the concept of "flatness" from the finite-dimensional to the infinite-dimensional systems through the parametrizations of trajectories via Gevrey functions and advance/delay operations on the reference trajectory.

Homework

1. For the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u_x(0) &= 0\,, \end{aligned}$$

find the input reference signal $u^r(1,t)$ so that the output $u(0,t)$ obeys the reference signal

$$u^r(0,t) = t^3\,.$$

2. For the heat equation

$$\begin{aligned} u_t &= u_{xx} \\ u(0) &= 0\,, \end{aligned}$$

find the input reference signal $u^r(1,t)$ so that the output $u_x(0,t)$ obeys the reference signal

$$u_x^r(0,t) = \sin \omega t\,.$$

3. For the Euler-Bernoulli beam

$$\begin{aligned} u_{tt} + u_{xxxx} &= 0 \\ u_{xx}(0) &= 0 \\ u_{xxx}(0) &= 0\,, \end{aligned}$$

show that

$$u^r(x,t) = \frac{\sinh\left(\sqrt{\omega}x\right) + \sin\left(\sqrt{\omega}x\right)}{2\sqrt{\omega}} \sin(\omega t)$$

is a solution to the system. This result shows that you can produce the output trajectory

$$\begin{aligned} u^r(0,t) &= 0 \\ u^r_x(0,t) &= \sin(\omega t) \end{aligned}$$

with the controls

$$\begin{aligned} u^r(1,t) &= \frac{\sinh\sqrt{\omega} + \sin\sqrt{\omega}}{2\sqrt{\omega}} \sin(\omega t) \\ u^r_x(1,t) &= \frac{\cosh\sqrt{\omega} + \cos\sqrt{\omega}}{2} \sin(\omega t)\,. \end{aligned}$$

4. Consider the undamped wave equation

$$\begin{aligned} u_{tt} &= u_{xx} \\ u_x(0) &= 0\,. \end{aligned}$$

Find the state reference trajectory $u^r(x,t)$ that corresponds to the output reference

$$u^r(0,t) = \sin\omega t\,.$$

Then, recalling that

$$u_x(1,t) = -c_0 u(1,t) - c_1\left(u_t(1,t) + c_0\int_0^1 u_t(y,t)dy\right), \qquad c_0, c_1 > 0$$

is a stabilizing controller, find the functions $M(\omega, c_0, c_1)$ and $\phi(\omega, c_0, c_1)$ to ensure that the controller

$$\begin{aligned} u_x(1,t) &= M(\omega, c_0, c_1)\sin\left(\omega t + \phi(\omega, c_0, c_1)\right) \\ &\quad - c_0 u(1,t) - c_1\left(u_t(1,t) + c_0\int_0^1 u_t(y,t)dy\right) \end{aligned}$$

guarantees that the output $u(0,t)$ achieves asymptotic tracking of the output reference $u^r(0,t) = \sin\omega t$.

Hint: First show that

$$
\begin{aligned}
M(\omega, c_0, c_1)\sin(\omega t + \phi(\omega, c_0, c_1)) &= u_x^r(1,t) + c_0 u^r(1,t) \\
&\quad + c_1\left(u_t^r(1,t) + c_0\int_0^1 u_t^r(y,t)dy\right).
\end{aligned}
$$

Note that the left side of this expression is much more suitable for online implementation because M and ϕ can be precomputed, whereas the implementation on the right requires the integration of the reference signal to be done online.

5. This exercise presents an alternative approach to doing trajectory tracking, when compared to the approach in Example 8. Consider the reaction-diffusion equation

$$
\begin{aligned}
u_t &= u_{xx} + \lambda(x)u \\
u_x(0) &= 0\,.
\end{aligned}
$$

When $\lambda(x)$ is spatially varying, the motion planning procedure cannot produce a closed-form solution, even for basic output trajectories like $u^r(0,t) = \sin\omega t$ or $u^r(0,t) = \mathrm{e}^{\alpha t}$. (In fact, even for the case $\lambda = \text{const} \neq 0$, the reference trajectory becomes considerably more complicated than the trajectory for $\lambda = 0$.) However, if our objective is just tracking, namely not trajectory generation per se but finding a feedback law that stabilizes a trajectory that corresponds to a certain output reference, then it turns out that the feedback law

$$
\begin{aligned}
u(1,t) &= \frac{1}{2}\left[\mathrm{e}^{\sqrt{\frac{\omega}{2}}}\sin\left(\omega t+\sqrt{\frac{\omega}{2}}\right)+\mathrm{e}^{-\sqrt{\frac{\omega}{2}}}\sin\left(\omega t-\sqrt{\frac{\omega}{2}}\right)\right] \\
&\quad + \int_0^1 k(1,y)u(y,t)dy,
\end{aligned}
$$

where $k(x,y)$ is the solution of the kernel PDE

$$
\begin{aligned}
k_{xx}(x,y)-k_{yy}(x,y) &= \lambda(y)k(x,y) \\
k_y(x,0) &= 0 \\
k(x,x) &= -\frac{1}{2}\int_0^x \lambda(y)dy,
\end{aligned}
$$

guarantees that

$$
\begin{aligned}
u(x,t) &- \int_0^x k(x,y)u(y,t)dy \\
&-\frac{1}{2}\left[\mathrm{e}^{\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t+\sqrt{\frac{\omega}{2}}x\right)+\mathrm{e}^{-\sqrt{\frac{\omega}{2}}x}\sin\left(\omega t-\sqrt{\frac{\omega}{2}}x\right)\right]\to 0
\end{aligned}
$$

as $t \to \infty$ for all $x \in [0,1]$, which means, in particular, that asymptotic tracking of the reference output $u^r(0,t) = \sin \omega t$ is achieved, namely,

$$u(0,t) - \sin \omega t \to 0\,, \quad \text{as } t \to \infty\,.$$

Explain (prove) this result. It is helpful to use the following notation:

$$\begin{aligned} w(x,t) &= u(x,t) - \int_0^x k(x,y)u(y,t)dy \\ w^r(x,t) &= u^r(x,t) - \int_0^x k(x,y)u^r(y,t)dy \\ \tilde{w}(x,t) &= w(x,t) - w^r(x,t) \end{aligned}$$

and note that all the three w-variables, w, w^r, and $\tilde{w}$, satisfy a heat equation with a Neumann boundary condition at $x = 0$ and with a boundary condition given by

$$\tilde{w}(1,t) = w(1,t) - w^r(1,t) = 0\,, \tag{551}$$

which determines the control law.

The point of this exercise is that trajectory tracking can be pursued for complicated, spatially-varying, parabolic and hyperbolic PDEs if one uses backstepping for trajectory stabilization. It is sufficient to develop trajectory generation for the heat or wave equation and add the input (labeled in this exercise as $w^r(1,t)$) to the usual stabilizing backstepping feedback law.